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Note on a nonlinearity parameter of surface waves

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Abstract

A nonlinearity parameter which is valid for both deep and shallow water waves is introduced. The parameter may be regarded as a wave Froude number. A modified form of the Ursell number is also discussed.

1. Introduction

The absence of a unique nonlinearity parameter for water waves in arbitrary depth obviously originates from the ever conflicting duality of deep water waves versus shallow water waves. The apparently unbridgeable gap between these two main bodies of work has led to basically two different nonlinearity parameters: $\varepsilon_d = ka$ and $\varepsilon_s = a/h$ where k is the wave-number, a is the wave amplitude, and h is the water depth. Here, through the use of nonlinear free surface conditions in conjunction with the results of linear theory, a nonlinearity parameter is derived and shown to converge to both ε_d and ε_s in the appropriate limits.

2. Derivation

In order to establish a non-dimensionalization procedure we first note that for waves propagating in water of arbitrary depth, linear theory predicts that orbital velocities are proportional to ga/c where g is the gravitational acceleration, a is wave amplitude and $c = [(g/k)\tanh kh]^{1/2}$ is phase speed. Surface elevation is simply proportional to wave amplitude a. Spatial variables may be scaled with wave-number, and a meaningful time scale would be the time necessary for a wave to travel a distance of its own length. In mathematical terms, we have

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$$(x,y,z)=k(\bar{x},\bar{y},\bar{z}), \quad t=kc\bar{t}, \quad \eta=\bar{\eta}/a$$

$$(u,v,w)=(c/ga)(\bar{u},\bar{v},\bar{w}), \quad \phi=(kc/ga)\bar{\phi}$$

where the dimensional variables are marked by an overbar and it has tacitly been assumed that a velocity potential does exist.

The equations governing the motion of surface waves are Laplace's equation in the domain, bottom condition, and kinematic and dynamic boundary conditions on the free surface. When the non-dimensionalization procedure suggested above is carried out, Laplace's equation and bottom condition remain unchanged while the free surface conditions become

$$\frac{\partial\eta}{\partial t}+\epsilon\frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x}+\epsilon\frac{\partial\phi}{\partial y}\frac{\partial\eta}{\partial y}=\frac{\epsilon}{\epsilon_d}\frac{\partial\phi}{\partial z}, \quad z=\epsilon_d\eta \tag{1}$$

$$\eta+\frac{\partial\phi}{\partial t}+\frac{\epsilon}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right]=0, \quad z=\epsilon_d\eta \tag{2}$$

where the nonlinearity parameter ε is given by

$$\epsilon=\frac{ga}{c^2} \tag{3}$$

which may be regarded as a *wave Froude number*. The nonlinearity parameter defined above embodies both ε_d and ε_s as special cases. For deep water waves c^2 approaches to g/k hence ε becomes $\varepsilon_d=ka$. On the other hand, in shallow water we have $c^2=gh$, which, when used in Eq. (3), gives $\varepsilon=\varepsilon_s=a/h$. Theoretical significance of such a unified parameter is discussed by Beji and Nadaoka (1995) for a nonlinear wave model which is applicable at arbitrary depths. Note that in essence $\varepsilon=u/c$, the ratio of particle velocity to phase speed (Froude number), which indicates that although both ε_d and ε_s appear as the measures of wave steepness, ε conveys the additional connotation of being a measure of relative velocities. This point is most relevant in obtaining Miche's breaking criteria which is considered later.

3. Remarks on wave Froude number and Ursell number

In order to elucidate the essential aspects of the nonlinearity parameter introduced we shall consider a demonstrative example. Suppose that a unidirectional wave of arbitrary amplitude and period propagates over a uniform slope. For such a case the use of linear theory for estimating the increase or decrease in amplitude and wave-number is a plausible approximation and allows us to compute the spatial variations of ε_d, ε_s, and ε easily. In Fig. 1 these nonlinearity parameters are compared for three different incident wave periods, corresponding respectively to intermediate $kh=\pi/2$, deep $kh=3\pi/2$, and shallow $kh=\pi/10$ water waves. The slope is 1 : 50 and the water depth in the deepest region is 25 meters, which reduces to 5 meters after a distance of 1000 meters. The incident wave amplitude is taken as 1 meter for simplicity. In case of the intermediate water wave, neither ε_d nor ε_s is

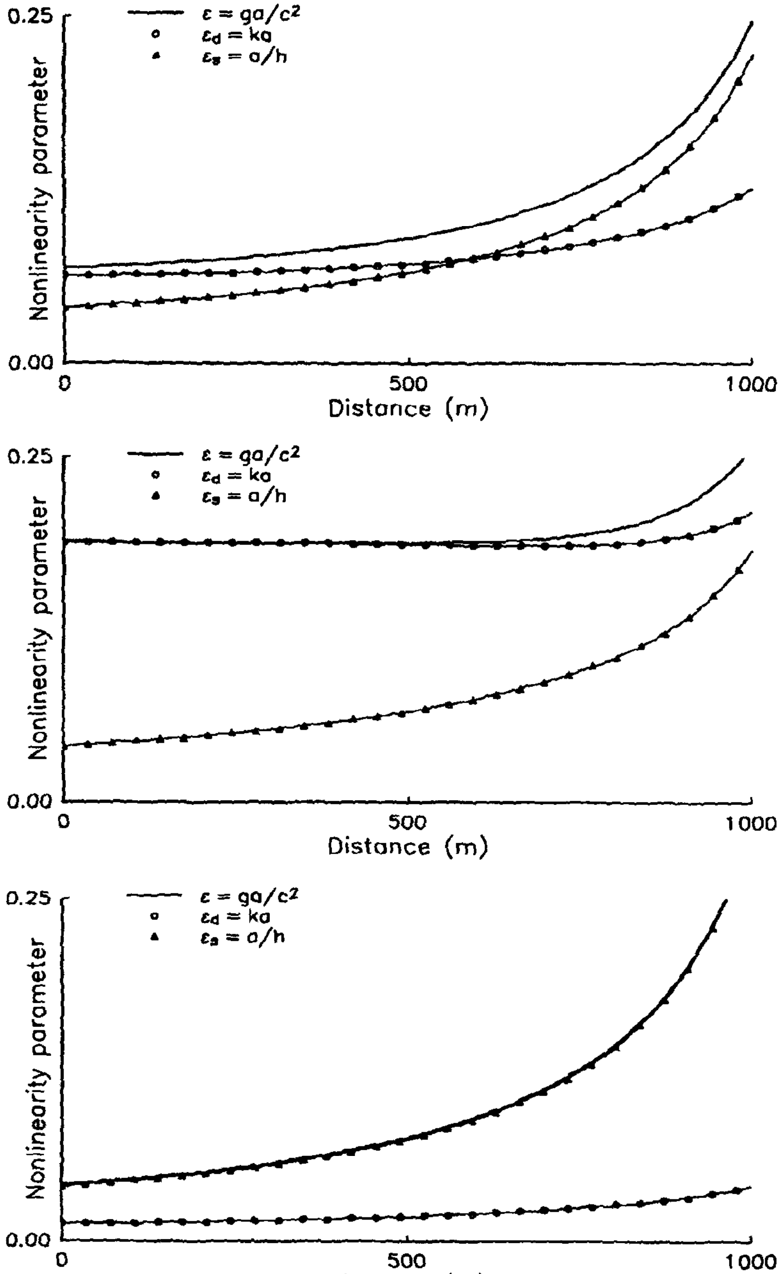

Fig. 1. Spatial variations of the nonlinearity parameters ε, ε_d and ε_s for a uniform slope of 1 : 50. The incident wave amplitude is 1 meter at water depth of 25 meters. In the upper graph the incident wave is intermediate-water wave with $kh = \pi/2$. The middle graph shows the case for deep-water $kh = 3\pi/2$, and finally the lower graph is the shallow-water case $kh = \pi/10$.

a good approximation to ε, which is presumably a better definition. Note ε_d in the deeper region (0–200 m) and ε_s in the shallower region (800–1000 m) are comparable with ε, as quite expected. The graphs showing deep $kh = 3\pi/2$ and shallow $kh = \pi/10$ water cases reveal at once that in the former ε_d and in the latter ε_s is an acceptable substitute for ε, though some divergence is observed beyond the validity range of ε_d. In both cases however ε_d and ε_s strongly disagree with each other, demonstrating their separate domains of validity. The overall aspect of these comparisons shows that in comparison with ε, ε_d and ε_s always provide lower estimates for nonlinearity.

A further point to note is the ratio $\varepsilon/\varepsilon_d$ appearing in Eq. (1). In deep water this ratio tends to unity and indicates that $\partial\eta/\partial t$ and $\partial\phi/\partial z$ are of the same order of magnitude. In shallow water however, the same ratio goes to $1/kh$ (or $1/\mu$ as defined in the context of shallow water theory) and indicates that the non-dimensional vertical velocity must be scaled up by an amount of $1/\mu$ to balance the temporal variation of surface. This in turn implies the well-known fact that the magnitude of vertical particle velocities are smaller in shallow water than those in deep water. The fact that $\varepsilon/\varepsilon_d$ converges to $1/\mu$ for shallow water waves may well lead us to recognize $\varepsilon_d/\varepsilon$ as a new definition of the dispersion parameter μ. Then, by establishing an analogy with the classical definition of the Ursell number, $Ur = \varepsilon_s/\mu^2$, it is possible to define a variant of this parameter in terms of the ε and $\varepsilon_d/\varepsilon$ as $\varepsilon/(\varepsilon_d/\varepsilon)^2$ which is $ka/(\tanh kh)^3$. For small kh this definition becomes identical with classical parameter, $Ur = a/k^2h^3$, as indicated. On the other hand, for large kh it tends to ka while the Ursell number vanishes. Since the Ursell number is introduced to resolve the shallow water wave conflict (Ursell, 1953) its value for large kh is not quite meaningful. In this respect, unlike the new nonlinearity parameter, the new definition of the Ursell number does not provide a significant contribution. Nonetheless, for intermediate-shallow water waves the quantitative estimate of $ka/(\tanh kh)^3$ is probably better than that of the original definition. Indeed it can be easily observed that (considering the deep water limit) the Ursell number always gives a lower estimate in comparison with $ka/(\tanh kh)^3$.

4. Miche's breaking criteria

Following an intuitive approach Miche postulated that breaking occurs when the fluid particle velocity at the wave crest equals the phase speed. Using the Stokes wave solution he calculated a theoretical limit for the maximum wave height (Mei, 1989, p. 469):

$$kH_{\max} = 0.88\tanh(kh) \tag{4}$$

where $H_{\max}$ is the maximum wave height possible at the onset of breaking.

Recalling that Eq. (3) may be written as $\varepsilon = u/c$ it becomes at once obvious that this parameter is in close correspondence with Miche's breaking criteria. Thus, supposing that at a certain maximum value of ε, say $\varepsilon = \beta$, breaking occurs and substituting $c^2 = (g/k)\tanh(kh)$ into Eq. (3) and re-arranging give $ka = \beta\tanh(kh)$, which, when $a = H_{\max}/2$ and $\beta = 0.44$, becomes identical with Miche's breaking criteria. Of course this simple demonstration is not the actual derivation of Miche's criteria; the main purpose here is to establish the relevance of the new parameter to actual physics.

Finally, as the parameter introduced here is nothing but a Froude number, it is a simple matter to show that the same parameter may also be obtained from Euler's equations of motion via the non-dimensionalization procedure used here.

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