NEW APPROACHES FOR COMPUTING WAVE GROWTH RATE DUE TO WIND INDUCED SHEAR INSTABILITIES

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New approaches for computing the wave growth rate due to wind induced shear instabilities are presented. The first approach is based on an analytical solution of Rayleigh's equation for arbitrary wind profiles in the vicinity of the critical point, followed by numerical integration. The wave growth rate is obtained from the dispersion relation of the air-sea interface. The results of the first approach agree perfectly well with the numerical solution of Conte and Miles (1959) for the special case of a logarithmic wind profile. The second approach assumes a definite vertical profile for the perturbed velocity field and then makes use of the air-sea dispersion relation for computing the wave growth rate. Despite the simplicity of the approximation the agreement of the second approach with the first one is quite acceptable. Comments on the future work are given in closing.

Keywords: Wind waves; shear instabilities; wave growth; Rayleigh equation.

1. Introduction

In studying the instability of sensitive jets Rayleigh (1880) suggested an improvement in the theory by supposing a gradual change in velocity and thus proceeded to derive an equation which is known today as the Rayleigh equation. Miles (1957) proposed a model for the growth of wind waves on the basis of Rayleigh's equation. Later, Conte and Miles (1959) gave accurate computations of wave growth rates by numerical solution of Rayleigh's equation for a *logarithmic* wind profile. This work first considers a different approach of solving the Rayleigh equation for *arbitrary* mean wind profiles by implementing the ideas of Rayleigh, which render the equation analytically solvable in the immediate vicinity of the singular point hence providing the initial values for the numerical integration. Furthermore, the wave growth rate is obtained from the dispersion relation of the air-water interface, which involves the vertical integration of the disturbed vertical velocity. The growth rates obtained are then compared with those of Conte and Miles (1959) and found to be in excellent agreement. As a second approach, a definite vertical profile is assumed for the velocity field and then the wave growth rate is computed from the dispersion relation again. Acceptable results are obtained although not as good as the first approach. In closing, comments on the development of a vertically integrated model of coupled air-water system are made.

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2. Rayleigh Equation and Its Approximate Solution

In two dimensions the governing equations of linearized perturbed shear flow with a prescribed mean wind velocity $U(z)$ are given by

$$u_t + U(z)u_x + U'(z)w = -p_x/\rho_a, \tag{1}$$

$$w_t + U(z)w_x = -p_z/\rho_a - g, \tag{2}$$

$$u_x + w_z = 0, \tag{3}$$

where u and w are horizontal and vertical velocity components respectively, p is the pressure, ρ_a is the air density, and g is the gravitational acceleration. Subscripts indicate partial differentiation with respect to the indicated variable and prime indicates differentiation with respect to z.

Assuming the horizontal motion periodic in time and space the disturbed vertical velocity component is taken as

$$w(x, z, t) = W(z) \exp ik(x - ct) \tag{4}$$

where k is the wavenumber, c is the wave celerity, and $W(z)$ is a function of the vertical coordinate z only. From the continuity equation,

$$u(x, z, t) = (i/k)W'(z) \exp ik(x - ct). \tag{5}$$

Eliminating pressure by cross differentiating (1) and (2) and making use of (4) and (5) result in the Rayleigh equation in terms of $W(z)$:

$$[U(z) - c]\,(W'' - k^2W) - U''(z)W = 0. \tag{6}$$

The above equation is obviously singular at the critical height $z = z_c$ where $U(z_c) = c$. At present no analytical solution of the Rayleigh equation exists; therefore, it is the usual approach to resort to a combination of analytical and numerical methods (see Miles, 1957 and Conte and Miles, 1959). Here, based on Rayleigh's (1895) ideas, an unconventional approach is adopted for obtaining an approximate analytical solution of equation (6) around the singular point z_c for an arbitrary wind profile $U(z)$. Accordingly, in the vicinity of z_c the mean wind profile $U(z)$ and its second derivative $U''(z)$ are approximated as

$$U(z) \simeq U'(z_c)(z - z_c) + c, \qquad U''(z) \simeq U''(z_c). \tag{7}$$

Using these approximations in equation (6) results in the following differential equation which is valid expressly in the close neighborhood of z_c:

$$W'' + \left[\frac{1}{\tilde{z}} - k^2\left[\frac{U'(z_c)}{U''(z_c)}\right]^2\right] W = 0 \tag{8}$$

in which $\tilde{z} = -U''(z_c)(z - z_c)/U'(z_c)$. Near the critical point $\tilde{z}$ approaches zero hence $1/\tilde{z}$ becomes large in comparison with $[kU'(z_c)/U''(z_c)]^2$ and equation (8) may be further approximated as

$$W'' + \frac{1}{\tilde{z}}W = 0 \tag{9}$$

which is a Riccati equation. First changing the dependent variable $W = \sqrt{\tilde{z}}\Psi$ and then changing the independent variable $\xi = 2\sqrt{\tilde{z}}$ transform equation (9) to

$$\xi^2\Psi_{\xi\xi} + \xi\Psi_\xi + (\xi^2 - 1)\Psi = 0, \tag{10}$$

which is a Bessel equation of order one. The two linearly independent solutions of equation (10) are given in terms of the Bessel functions of the order one:

$$\Psi(\xi) = AJ_1(\xi) + BY_1(\xi), \tag{11}$$

where A and B are arbitrary constants. Changing to the original variables gives

$$W(\tilde{z}) = \sqrt{\tilde{z}}\left[AJ_1(2\sqrt{\tilde{z}}) + BY_1(2\sqrt{\tilde{z}})\right], \tag{12}$$

in which $\tilde{z} = -U''(z_c)(z - z_c)/U'(z_c)$ as defined before. Note that for negative values of $\tilde{z}$ the argument of the Bessel functions is pure imaginary.

The above solution, which is valid around the singularity, provides the initial values for the numerical integration of equation (6). The details of the numerical procedure may be described as follows. First, a small quantitity, say $\varepsilon \sim 10^{-3} - 10^{-6}$, is selected and just below the critical point $z_{\varepsilon_-} = z_c(1 - \varepsilon)$ is defined. Then, $\sqrt{\tilde{z}}J_1(2\sqrt{\tilde{z}})$ and $[\sqrt{\tilde{z}}J_1(2\sqrt{\tilde{z}})]'$ evaluated at $\tilde{z}_{\varepsilon_-} = -U''(z_c)(z_{\varepsilon_-} - z_c)/U'(z_c)$ supply the starting values for the first linearly independent solution while $\sqrt{\tilde{z}}Y_1(2\sqrt{\tilde{z}})$ and $[\sqrt{\tilde{z}}Y_1(2\sqrt{\tilde{z}})]'$ for the second solution. For the present problem $\tilde{z}_{\varepsilon_-} < 0$, therefore care must be observed in using the complex conjugate values of $\sqrt{\tilde{z}}Y_1(2\sqrt{\tilde{z}})$ and $[\sqrt{\tilde{z}}Y_1(2\sqrt{\tilde{z}})]'$ at $\tilde{z}_{\varepsilon_-}$ for ensuring a positive growth rate since these terms implicity contain the logarithmic singularity. Having prescribed the necessary starting values, equation (6) is numerically integrated in the negative direction (below the critical point) for each set of initial conditions between z_{ε_-} and z_0, which is termed the roughness length of the air-water interface. The values of linearly independent solutions at the lower limit z_0, as obtained from two parallel numerical integrations, are denoted by $W_{J_1}(z_0)$ and $W_{Y_1}(z_0)$, which are kept in memory for later use in satisfying the boundary conditions. The second part of the numerical integration proceeds in the positive direction (above the critical point) between $z_{\varepsilon_+} = z_c(1 + \varepsilon)$ and z_∞, which is a relatively large and indefinite upper limit to be determined according to a convergence criterium. The starting values are again provided from the analytical solution as described above but this time evaluated at $\tilde{z}_{\varepsilon_+} = -U''(z_c)(z_{\varepsilon_+} - z_c)/U'(z_c)$. At every integration step in the positive direction the boundary conditions are used to determine the unknown coefficients, say A and B again, of the desired solution. If the difference in one of the coefficients between two successive steps is less than a specified small value the computation is terminated and the computed values of the linearly independent solutions are denoted by $W_{J_1}(z_\infty)$ and $W_{Y_1}(z_\infty)$. Trial computations show that instead of applying a convergence criterium with an unfixed upper limit of integration it is more convenient and quite sufficient to perform the integration to the fixed height of $\lambda = 2\pi/k$.

The boundary conditions imposed are typical to these kind of problems. Just above the interface at $z = z_0$, a definite value for the vertical velocity is enforced. For great heights, $z = z_\infty$, the disturbances are assumed to vanish. Accordingly,

$$W(z_0) = W_0, \qquad AW_{J_1}(z_0) + BW_{Y_1}(z_0) = W_0, \tag{13}$$

$$W'(z_\infty) + kW(z_\infty) = 0, \qquad A\left[W'_{J_1}(z_\infty) + kW_{J_1}(z_\infty)\right] + B\left[W'_{Y_1}(z_\infty) + kW_{Y_1}(z_\infty)\right] = 0. \tag{14}$$

Solving for the unknown coefficients A and B gives

$$A = \frac{-[W'_{Y_1}(z_\infty) + kW_{Y_1}(z_\infty)]W_0/W_{Y_1}(z_0)}{\{W'_{J_1}(z_\infty) + kW_{J_1}(z_\infty) - [W'_{Y_1}(z_\infty) + kW_{Y_1}(z_\infty)]W_{J_1}(z_0)/W_{Y_1}(z_0)\}}, B = \frac{[W_0 - AW_{J_1}(z_0)]}{W_{Y_1}(z_0)} \tag{15}$$

which are evaluated at every integration step in the positive $z-$direction till the specified criterium is met at the previously unknown height z_∞. Note that since the solution is complex the coefficients A and B are complex too.

3. Dispersion Relation of Air-Sea Interface

Substituting equation (4) into (2), supposing for the air pressure $p(x,z,t) = P_a(z)\exp ik(x-ct)$, and integrating from the air-water interface $\eta = a\exp ik(x-ct)$ to $+\infty$ give for $P_a(\eta)$

$$P_a(\eta) = P_0 - \rho_a g a + i\rho_a k \int_{z_0}^{+\infty} [U(z)-c]\, W(z)dz, \tag{16}$$

where P_0 is the atmospheric pressure at the surface and the lower limit of the integration has been set to z_0 instead of η, since the problem is linearized. For later purposes it is necessary to make use of the kinematic boundary condition at $z=\eta$ for air:

$$\eta_t + U(z)\eta_x = w \qquad \text{at} \qquad z=\eta. \tag{17}$$

Similar to equation (16), the above boundary condition is now evaluated at the roughness height $z=z_0$ instead of the actual free surface $z=\eta$. Noting that by definition the mean wind velocity $U(z)$ vanishes at $z=z_0$ it is possible to write from equation (17)

$$-ikca = W(z_0) = W_0. \tag{18}$$

Using (18) in (16) gives

$$P_a(\eta) = P_0 - \rho_a g a + \rho_a k c^2 a \frac{(k/c)}{W_0} \int_{z_0}^{\infty} [U(z)-c]\, W(z)dz, \tag{19}$$

which is the air pressure on the free surface due to the wind.

For the water wave motion it is fairly straightforward to show that for deep water waves the pressure on the free surface can be expressed as

$$P_w(\eta) = P_0 - \rho_w g a + \rho_w k c^2 a. \tag{20}$$

The dispersion relation of the combined air-water system can be obtained from the continuity of pressure across the interface; that is, $P_a(\eta) = P_w(\eta)$:

$$P_0 - \rho_a g a + \rho_a k c^2 a \frac{(k/c)}{W_0} \int_{z_0}^{\infty} [U(z)-c]\, W(z)dz = P_0 - \rho_w g a + \rho_w k c^2 a \tag{21}$$

Eliminating P_0's, dividing by $\rho_w a$, and solving for c^2 result in

$$c^2 = (g/k)\left[(1-s)/(1-sI_c)\right], \tag{22}$$

where $s = \rho_a/\rho_w$ and the complex integral I_c is given by

$$I_c = \frac{(k/c)}{W_0} \int_{z_0}^{+\infty} [U(z)-c]\, W(z)dz. \tag{23}$$

Noting that $s \simeq 10^{-3}$ is a small quantity, equation (22) may be approximated as

$$c \simeq c_0[(1-s/2)/(1-sI_c/2)] \simeq c_0(1-s/2+sI_c/2), \tag{24}$$

in which $c_0 = \sqrt{g/k}$. Note that both $W(z)$ and c appearing in the integral I_c are complex; however, in evaluating the integral the unknown complex phase speed c may be taken approximately real as its imaginary part is relatively small. Once $W(z)$ is determined, the complex integral I_c hence the growth rate can be computed as the complex part of kc:

$$\gamma = k\Im(c) = (1/2)skc_0\Im(I_c), \tag{25}$$

where $\Im(I_c)$ denotes the imaginary part of the complex integral I_c. Miles (1957) defines a slightly different, non-dimensional growth rate β, which may be expressed in terms of $\Im(I_c)$ as

$$\beta = (c/U_1)^2 \Im(I_c). \tag{26}$$

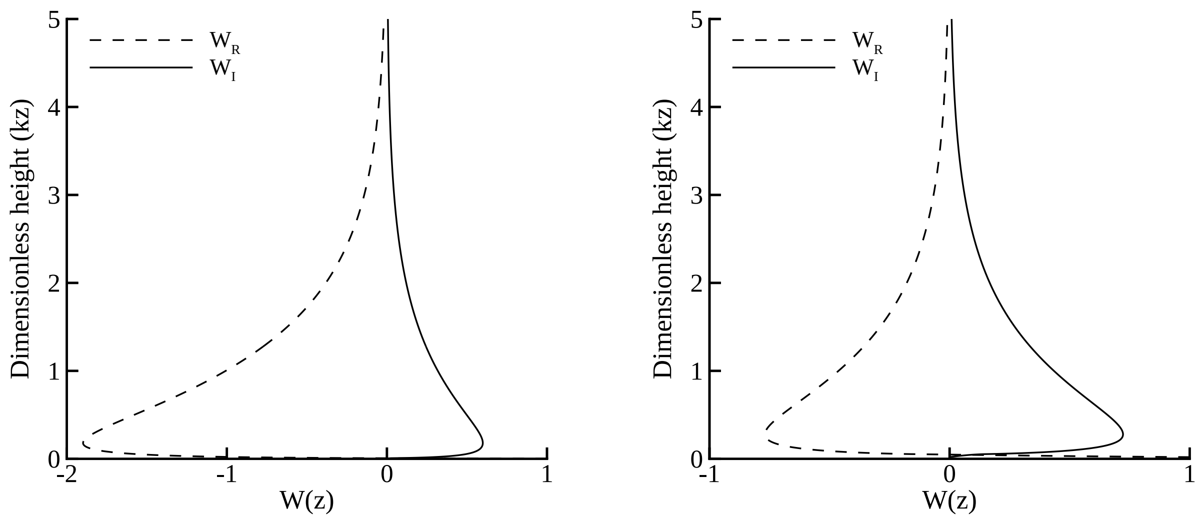
4. Computed Growth Rates and Velocity Profiles

Computations using the above described approach are compared with the results given by Conte and Miles (1959) for the growth rate β in Table 1 with excellent agreement.

	$\Omega = 3 \times 10^{-3}$		$\Omega = 1 \times 10^{-2}$		$\Omega = 2 \times 10^{-2}$	
c_0/U_1	Present work	Conte-Miles	Present work	Conte-Miles	Present work	Conte-Miles
1	3.533	3.536	3.233	3.237	2.744	2.747
2	3.412	3.414	3.298	3.302	2.925	2.928
3	3.431	3.433	3.205	3.208	2.775	2.779
4	3.428	3.431	2.962	2.966	2.424	2.427
5	3.297	3.301	2.544	2.547	1.907	1.909
6	2.971	2.975	1.963	1.965	1.287	1.288
7	2.438	2.441	1.289	1.290	0.677	0.677
8	1.748	1.750	0.647	0.646	0.224	0.223
9	1.015	1.016	0.193	0.193	0.026	0.026
10	0.405	0.405	0.018	0.018	0.00024	0.00024

Table 1: Comparisons of the dimensionless growth rate β for three different groups of wind parameters $\Omega = gz_0/U_1^2 = 3 \times 10^{-3}$ (left), $\Omega = 1 \times 10^{-2}$ (middle), and $\Omega = 2 \times 10^{-2}$ (right).

To give an idea about the nature of the solution, real and imaginary parts of the vertical air velocity $W(z)$ are shown in Figures 1a and 1b.



Figures 1a, b: Real and imaginary parts of the vertical dependency of the perturbed air velocity $W(z)$ for $c_0/U_1 = 1$ and $\Omega = 3 \times 10^{-3}$ (left), $\Omega = 2 \times 10^{-2}$ (right).

5. Approximate Vertical Profiles

While it is essential to have accurate means of computing the growth rates for *arbitrary* wind profiles as introduced in the previous section, approximate but more practical methods are needed, especially for developing vertically integrated and coupled wind-wave models. To this end, we first re-consider the approximate profile suggested by Miles (1957) for the vertical dependency of the disturbed velocity field and then introduce an improved version of it.

5.1 *Miles's Approximation*

Prior to the accurate numerical computations of Conte and Miles (1959), Miles (1957) made an attempt of computing the wave growth rates by an approximation. For the vertical dependency of the stream function Miles (1957) assumed a form, which is equivalent to the following expression of $W(z)$ for the vertical velocity $w(x,z,t) = W(z)\exp ik(x-ct)$:

$$W(z) = ika[U(z) - c]e^{-kz}, \qquad (27)$$

which satisfies the kinematic boundary condition, equation (17), at $z = z_0$ provided that e^{-kz_0} is negligibly small. Miles (1957) obtained the growth rates by means of approximate evaluation of a definite integral expression. Here, the vertical profile suggested by Miles, equation (27), is used in equations (23) and (26) respectively for computing the growth rate β; the integral appearing in (23) is evaluated numerically. Except for small c/U_1 values, for which Miles's estimates are lower, the results are in accord with Miles's values. However, when compared with the accurate numerical solution of Conte and Miles (1959), Miles's approximate method grossly overestimates the rates, amounting to as much as 6 times the correct value. Table 2 below makes the same comparisons given in Table 1 but this time between the approximate solution devised from equation (27) and the numerical solution of Conte and Miles (1959).

	$\Omega = 3\times10^{-3}$		$\Omega = 1\times10^{-2}$		$\Omega = 2\times10^{-2}$	
c_0/U_1	Miles's Ap.	Conte-Miles	Miles's Ap.	Conte-Miles	Miles's Ap.	Conte-Miles
1	19.60	3.54	10.87	3.24	7.13	2.75
2	22.98	3.41	13.30	3.30	9.02	2.93
3	21.26	3.43	12.02	3.21	7.98	2.78
4	14.66	3.43	9.43	2.97	5.96	2.43
5	13.52	3.30	6.58	2.55	3.83	1.91
6	9.51	2.98	4.02	1.97	2.05	1.29
7	6.02	2.44	2.06	1.29	0.85	0.68
8	3.30	1.75	0.80	0.65	0.23	0.22
9	1.46	1.02	0.20	0.19	0.03	0.03
10	0.45	0.41	0.02	0.02	0.00	0.00

Table 2: Comparisons of the dimensionless growth rate β for three different groups of wind parameters $\Omega = gz_0/U_1^2 = 3\times10^{-3}$ (left), $\Omega = 1\times10^{-2}$ (middle), and $\Omega = 2\times10^{-2}$ (right).

The overestimated growth rates clearly indicate that the functional dependency of $W(z)$ as given by (27) is improper. Actually, it is possible to show that such a form is more suitable for the real part of the solution of the Rayleigh equation. Since the imaginary solution determines the growth rate, a modified form of (27) is suggested in the following.

5.2 *An Improved Approximation*

Miles's approximation serves as a good starting point for selecting a better vertical profile for the disturbed velocity. After examining the general characteristics of the imaginary solutions as obtained from the accurate numerical solutions, Miles's approximate profile is modified as

$$W(z) = ika[U(z) - c](z_0/z)^{\alpha} e^{-k(z-z_0)}, \tag{28}$$

where α is a free parameter to be determined. Note that the above profile satisfies the kinematic boundary condition at $z = z_0$ without any restriction so that $W(z_0) = -ikca$ as in (18).

In principle α should be determined from the appropriate governing equation by imposing a definite technique of minimizing a definite error such as using the least squares technique for minimizing the integrated error with respect to α. However, since c is imaginary, such an approach requires the simultaneous solutions of the Rayleigh equation for the real and imaginary parts and equation (23) for c. For the time being we shall not pursue this course and, based on numerical experiments, propose an empirical expression for α as

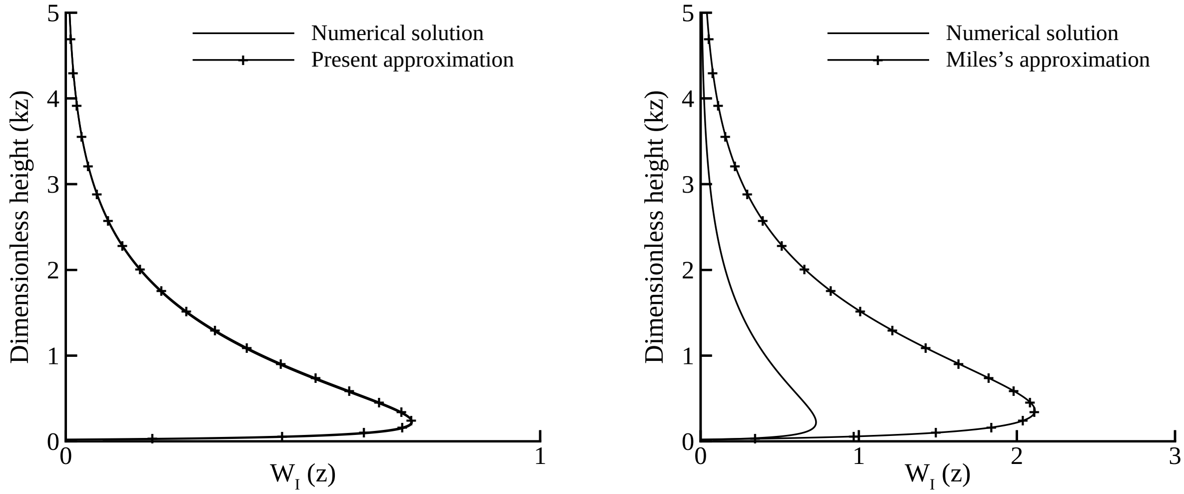
$$\alpha = \left[e^{-1.565\Omega^{0.12875}} - 0.00552 \left[1 + 3.3\sqrt{\Omega} - 2.1\sqrt[3]{\Omega} \right] (c_0/U_1) \right] / \sqrt[3]{3}. \tag{29}$$

For a wide range of c_0/U_1 and Ω the parameter α is a positive number less than 0.3. If the above formula produces an α value less than zero α is set to zero, which means the above model becomes identical with Miles's approach. Such a condition occurs only for relatively high c_0/U_1 values for which Miles's approach becomes good as well. Table 3 presents the comparisons of the dimensionless growth rate β between the present approximate method and the numerical solution of Conte and Miles (1957). The largest error is 6.7% for $c_0/U_1 = 6$ and $\Omega = 3 \times 10^{-3}$. Numerical computations involving much wider Ω values revealed that equations (28) and (29) still give reliable results with errors not exceeding 7% or so.

	$\Omega = 3 \times 10^{-3}$		$\Omega = 1 \times 10^{-2}$		$\Omega = 2 \times 10^{-2}$	
c_0/U_1	Present Ap.	Conte-Miles	Present Ap.	Conte-Miles	Present Ap.	Conte-Miles
1	3.61	3.54	3.25	3.24	2.72	2.75
2	3.58	3.41	3.44	3.30	3.04	2.93
3	3.46	3.43	3.24	3.21	2.81	2.78
4	3.31	3.43	2.89	2.97	2.38	2.43
5	3.10	3.30	2.42	2.55	1.84	1.91
6	2.78	2.98	1.86	1.97	1.24	1.29
7	2.33	2.44	1.23	1.29	0.66	0.68
8	1.74	1.75	0.64	0.65	0.23	0.22
9	1.08	1.02	0.20	0.19	0.03	0.03
10	0.45	0.41	0.02	0.02	0.00	0.00

Table 3: Comparisons of the dimensionless growth rate β for three different groups of wind parameters $\Omega = gz_0/U_1^2 = 3 \times 10^{-3}$ (left), $\Omega = 1 \times 10^{-2}$ (middle), and $\Omega = 2 \times 10^{-2}$ (right).

In order to demonstrate the accuracy of the vertical profile selected the present approximation for $W(z)$ and Miles's approximation are separately compared with the numerical solution in Figures 2a and 2b, respectively.



Figures 2a, b: Imaginary part of the vertical dependency of the perturbed air velocity as approximated by two different profiles compared with the numerical solution for $c_0/U_1 = 3$ and $\Omega = 1{\times}10^{-2}$.

Figure 2a further confirms that not only the integral value of the approximated profile but also the shape of the profile itself agrees with the accurate numerical solution of the Rayleigh equation and that the improved form proposed by equation (28) is a promising one for modelling the vertical dependency of the disturbed air velocity due to shear instabilities between the air-sea interface.

6. Concluding Remarks

Two new approaches are introduced for the computation of wave growth rate due to shear instabilities. The first method is based on Rayleigh's ideas of certain approximations in the vicinity of the critical point z_c where the mean wind velocity is equal to the wave velocity. The method is valid for *arbitrary* mean wind profiles and gives very accurate results as demonstrated for the special case of a logarithmic wind profile as adopted by Miles (1957) and Conte and Miles (1959). The second approach aims at developing a vertically integrated model for the more general problem of the coupled wind-wave problem and proposes a vertical distribution function for the disturbed air velocity. The profile introduced is seen to give quite acceptable results for the wave growth rates and expected to be set on firmer grounds by determining the free parameter α by an analytical approach instead of the empirical formula used here. The future work will focus on this particular problem first and then proceed to develop a vertically-integrated coupled wind-wave model.

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