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AUTHORS' REPLY TO 'DISCUSSION OF SCHÄFFER AND MADSEN ON "A FORMAL DERIVATION AND NUMERICAL MODELLING OF THE IMPROVED BOUSSINESQ EQUATIONS FOR VARYING DEPTH" (*Ocean Engineering*, **25** (1998) 497–500)'

Serdar Beji* and Kazuo Nadaoka†
*Department of Naval Architecture and Ocean Engineering, Istanbul Technical University, Maslak 80626, Istanbul, Turkey
†Graduate School of Information Science and Engineering, Tokyo Institute of Technology, 2-12-1 O-okayama, Meguro-ku, Tokyo 152, Japan

The discussion of Schäffer and Madsen is appreciated for providing an opportunity to clarify some definite points which were only briefly mentioned in Beji and Nadaoka (1996). More importantly, their inquiries have led us to new conclusions regarding Boussinesq equations in general and Nwogu's equations (Nwogu, 1993) in particular. It is therefore expected that most of the questions about all these equations may now be settled satisfactorily.

1. PARTIAL REPLACEMENT VERSUS ADDITION

We would first like to clarify the fundamental differences between BN's partial replacement method and MS's addition method (Madsen and Sørensen, 1992). It must be emphasized that the formalism of the two approaches is not equivalent; there is a subtle but very distinct difference between the two. Indeed, if they were the same the results would not differ in the first place.

BN's procedure leads to a *single* possible form for the improved model, dictated entirely by the rules of mathematical physics. The model *cannot* be shaped beforehand according to particular aims; the outcome is, just like any formal derivation, determined by the formalism alone.

MS's procedure, on the other hand, allows *any* possible form for the improved model as long as the added terms are second-order and dimensionally correct. Thus, the model *can* be shaped beforehand according to specific choices. The model of Schäffer and Madsen (1995) is the ultimate product of this approach, from which *any* Boussinesq model is recovered with the further aid of the shoaling parameter, which compensates for the differences arising in the linear shoaling terms. It is quite plausible to argue the merits and flexibilities of such a formalism, yet not guided by a completely binding rule, one is compelled to make heuristic choices. Since any possible choice is also an acceptable choice, the quest for a single formal model is rendered obsolete. This point is in opposition to the basic approach of BN, and hence shows the essential difference between the two.

2. CONSERVATION OF ENERGY FLUX

Schäffer and Madsen argue that, while some Boussinesq models conserve the energy flux exactly, some do not, and that the BN model belongs to the former, while the MS and Nwogu models belong to the latter. Further, they conclude that since Nwogu's model is a perfectly formal derivation, there could be no definite connection between the formalism and the conservation of energy flux. After a re-assessment of Nwogu's model, we do agree that his model does not conserve the energy flux perfectly, but that there is still a definite connection between the formalism and energy flux conservation.

Nwogu's equations are indeed a challenge to combine. However, after a protracted algebra and neglect of a definite linear shoaling term $\propto\tilde{u}_{xxxxt}$ as a higher-order contribution the authors had previously arrived at

$$\eta_{tt} - gh\eta_{xx} - \left(\alpha + \frac{1}{3}\right)gh^3\eta_{xxxx} + \alpha h^2\eta_{xxtt}$$
$$= h_x\left[g\eta_x - 3\alpha h\eta_{xtt} + 6\left(\alpha + \frac{1}{3}\right)gh^2\eta_{xxx}\right]$$

where α is the dispersion parameter of Nwogu. Setting $\alpha = -(1 + \beta)/3$ as dictated by the correspondence between Nwogu's model and BN's model, one obtains exactly the combined form of BN's model (eqn. (8)). This was the reason that led the authors to state that Nwogu's model also conserves energy flux.

Considering the fact that no such term had to be neglected in deriving BN's combined model, and presuming that the new approach of Schäffer and Madsen (1995) for obtaining the shoaling gradient of Nwogu's model is exact within the usual approximations, it must be admitted that the neglect of the term $\propto\tilde{u}_{xxxxt}$ is not justified, that the above equation is merely the result of a fortuitous coincidence, and that Nwogu's model does not conserve the energy flux.

We are facing the rather curious fact that, while the Boussinesq models in terms of the depth-averaged velocity and surface velocity (and their formal variations, which include BN's model) conserve the energy flux perfectly, all others fail to do so. What is then the cause of this difference between these two types of equations? A very plausible explanation is as follows.

The Boussinesq equations in terms of the mean velocity and the surface velocity ($z = 0$), as compared with the rest, have the important advantage that in the former the continuity equation and in the latter the momentum equation is *exact* to all orders of dispersion parameter. The truncation errors then exist only in either momentum or continuity equation, in contrast to all the other models which contain truncation errors both in continuity and momentum equations. For this reason the Boussinesq equations formulated in terms of the variables indicated above are more accurate than the rest, a fact further supported by energy considerations. Also, the formally manipulated forms of these equations, like BN's model, retain the same order of accuracy. For instance, it is possible to derive an equivalent form of BN's model in terms of the surface velocity $\mathbf{u}'$. Applying the partial replacement method to the equations of Peregrine (1967) results in

$$\mathbf{u}_t' + (\mathbf{u}' \cdot \nabla)\mathbf{u}' + g\nabla\eta = 0$$

$$\eta_t + \nabla[(h + \eta)\mathbf{u}'] + \frac{1}{2}(1 + \beta^*)\nabla\cdot\left\{h^2\nabla[\nabla\cdot(h\mathbf{u}')] - \frac{1}{3}h^3\nabla(\nabla\cdot\mathbf{u})\right\}$$

$$- \frac{1}{2}\beta^*\nabla\cdot\left[-\frac{2}{3}(h^2\nabla\eta_t + h\nabla h\eta_t)\right] = 0$$

which, in unidirectional form, may be combined (with the usual assumptions) as

$$\eta_{tt} - \mathrm{gh}\eta_{xx} = \frac{(1 + \beta^*)}{3}\mathrm{gh}^3\eta_{xxxx} + \frac{\beta^*}{3}\mathrm{h}^2\eta_{xxtt}$$
$$= \mathrm{h}_x[\mathrm{g}\eta_x - \beta^*\mathrm{h}\eta_{xtt} + 2(1 + \beta^*)\mathrm{gh}^2\eta_{xxx}]$$

Upon redefining the dispersion coefficient β^* as $-(1 + \beta)$, one obtains exactly eqn. (8) of BN, which has been shown to be consistent with energy requirements.

Nwogu's model, which contains truncation errors in both continuity and momentum equations, fails to meet energy requirements, as explicitly depicted by Schäffer and Madsen in their figure 2. MS's model, being derived from a more accurate Boussinesq model, is expected to conserve the energy flux. Its failure originates not from their starting equations but from the heuristic procedure they adopted. It may be remedied by following the partial replacement procedure, which yields the corresponding forms of BN's equations. However, if one is to abandon the question of energy conservation, the origin of errors becomes immaterial and MS's model may well be regarded as formal as Nwogu's model. After checking the dispersion characteristics of these two models, one is quite justified to choose MS's model.

3. LINEAR SHOALING CHARACTERISTICS

BN make no claim of the superiority of their model regarding the linear shoaling characteristics. These characteristics are dictated entirely by the formalism itself, which reveals that the linear shoaling characteristics are in essence determined by the linear dispersion characteristics and should not be manipulated independently. The emphasis of BN is on the formalism of their derivation procedure and the energy conservation characteristics of the resulting equations.

While the linear shoaling characteristics of MS's equations appear to be superior, before making a conclusive statement one must observe caution. It must be remembered that MS have abandoned their choice of the dispersion parameter $B = 1/21$ in favor of $B = 1/15$, which was previously deemed an *inferior* choice (Madsen *et al*., 1991). We do completely agree that, although $B = 1/21$ appears to provide a better fit to the linear theory dispersion relation than $B = 1/15$ does, the latter choice gives better computational results as observed by the first author here in his previous numerical experiments. Therefore, a similar caution ought to be observed in making a superiority declaration regrading the linear shoaling characteristics; further tests are necessary for a conclusive decision.

4. CLOSING REMARKS

The Boussinesq models in terms of the mean velocity (or depth-integrated velocity) and the surface velocity are recognized to be more accurate than all the others. The re-arranged versions of these equations (the BN model and the model derived above) are likewise shown to retain this high accuracy, which manifests itself as perfect energy conservation characteristics. Nwogu's model does not confirm the energy requirements due to the presence of higher-order truncation errors in both continuity and momentum equations. MS's model falls into this last category too, because of the terms embedded into the model by the addition process which, unlike Nwogu's model, may be corrected. If the question of energy conservation is regarded as important, the BN model and its equivalent in terms of the surface velocity as derived here are the models to choose from. Otherwise, all the models examined here are equally suitable.

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