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Applications of Rayleigh's instability equation

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Abstract

Rayleigh's instability equation is employed in the context of the water wave growth problem for different wind profiles. Following Rayleigh's (1880) approximations, solution of the singular Rayleigh equation for an arbitrary wind profile is devised in the form of a Riccati equation. Further, a vertically integrated air motion model is introduced for calculating growth rates. Two different wind profiles; namely, logarithmic and 1/7-power-law profiles, are used for comparing growth rates. Variations of the vertical air velocity by height are shown in graphs to provide visual aspects to the growth rate computations. In closing, new progress directions are commended.

1. Introduction

Assuming a gradual change in air velocity Rayleigh (1880) derived an equation for the instability of jets, which is known today as Rayleigh's instability equation or simply the Rayleigh equation. However, Rayleigh did not attempt to apply his theory to the problem of wave generation by wind.

Nearly 80 years later, Miles (1957) proposed a model for the growth of wind waves on the basis of Rayleigh's equation with a logarithmic wind profile that was in line with Rayleigh's assumption of a gradually changing air velocity. Miles's model predicts wave growth rates reasonably in agreement with the observations but underestimates the growth rate of long waves with phase speeds nearly equal to the wind speed. To improve the theory considering the effects of small-scale turbulence received most attention (Gent & Taylor 1976; Jakobs 1987); but results of these works did not differ much from those of Miles. A comprehensive review of the subject may be found in Miles (1997).

Although various aspects of the problem have been examined, almost universally accepted assumption of logarithmic mean wind profile has not been questioned at all. It is indeed the wind profile that basically determines the wave growth rates. Accordingly, Beji & Nadaoka (2004) introduced a different approach of solving Rayleigh's instability equation for *arbitrary* mean wind profiles. The solution technique, originally suggested by Rayleigh (1895) himself, was based on re-casting the Rayleigh equation into a form of Riccati equation. Then, wave growth rates were also computed in an unconventional way by implementing the dispersion relation of the air-sea interface, which involves the vertical integration of the disturbed vertical velocity weighted by the wind profile.

Here, by taking advantage of this solution technique, computations of growth rates for the 1/7-power-law wind profile are made and compared with those of the logarithmic profile. Variations of disturbed vertical air velocity by height are also presented to provide some physical insight into the nature of the process. In closing, possible new research directions on the subject are pointed out.

2. Rayleigh's equation and its solution

2.1. *Rayleigh's instability equation*

Rayleigh's instability equation for linearized inviscid shear flow of air over wavy water surface with a prescribed mean wind velocity $U(z)$ is given by

$$[U(z) - c\,]\,(W'' - k^2 W) - U''(z)W = 0 \tag{2.1}$$

where c is the wave celerity, k is the wavenumber, and $W(z)$ is a function of the vertical coordinate z only representing the vertical dependence of the disturbed vertical velocity component of air $w(x, z, t) = W(z)\exp[ik(x - ct)]$. The above equation is obviously singular at the critical height $z = z_c$ where $U(z_c) = c$. Analytical solution of the Rayleigh equation does not exist; therefore, it is the usual approach to resort to a combination of analytical and numerical methods.

2.2. *Solution approach*

Based on Rayleigh's (1895) ideas, Beji & Nadaoka (2004) developed an approach for obtaining an approximate analytical solution of equation (2.1) around the singular point z_c for an arbitrary wind profile $U(z)$ with non-zero second derivative. In the vicinity of the singular point z_c, the wind velocity profile $U(z)$ is approximated by its linearized form while the second derivative of $U(z)$ is replaced by its constant value at z_c. With the aid of these approximations and change of dependent variable, (2.1) is transformed into a Riccati type equation

$$\frac{d^2W}{d\tilde{z}^2} + \frac{1}{\tilde{z}}W = 0, \tag{2.2}$$

where $\tilde{z} = -U''(z_c)(z - z_c)/U'(z_c)$. The details of derivation can be found in Beji & Nadaoka (2004). The above approximate equation provides exact analytical solutions in the vicinity of the singular point z_c, which initiate the numerical integration of (2.1) above and below the singular point. Numerical treatment of the problem was first given by Conte & Miles (1959) and described explicitly in Beji & Nadaoka (2004).

2.3. *Growth rate*

Beji & Nadaoka (2004) obtained the wave growth rate in a different way by the use of the dispersion relation of the air-sea interface:

$$c^2 = \frac{g}{k}\frac{(1-s)}{\left[1 - s\frac{(k/c)}{W_0}\int_{z_0}^{+\infty}[U(z) - c]\,W(z)dz\right]}, \tag{2.3}$$

where $s = \rho_a/\rho_w$ is the ratio of air density to water density. Defining the complex integral,

$$I_c = \frac{(k/c)}{W_0}\int_{z_0}^{+\infty}[U(z) - c]\,W(z)dz, \tag{2.4}$$

and noting that $s \simeq 10^{-3}$ is a small quantity, (2.3) may be approximated as

$$c \simeq c_0[(1 - s/2)/(1 - sI_c/2)] \simeq c_0(1 - s/2 + sI_c/2), \tag{2.5}$$

in which $c_0 = \sqrt{g/k}$ is the deep water wave celerity. Note that both $W(z)$ and c appearing in the integral I_c are complex; but in evaluating the integral the unknown complex phase speed c may be taken approximately real as its imaginary part is proportional to s hence negligibly small. Once $W(z)$ is determined, the complex integral I_c hence the growth rate can be computed as the imaginary part of kc that would promote the growth (or decay) of the surface elevation:

$$\gamma = k\Im(c) = (1/2)skc_0\Im(I_c), \tag{2.6}$$

where $\Im(I_c)$ denotes the imaginary part of the complex integral I_c. Miles (1957) defines a slightly different, dimensionless growth rate β, which may be expressed in terms of the imaginary part of the complex integral I_c as

$$\beta = (c_0/U_1)^2 \Im(I_c), \tag{2.7}$$

where U_1 is a characteristic velocity related to the so-called friction velocity u_* by the relation $U_1 = u_*/\kappa$, κ being the von Kármán constant taken as $\kappa = 0.41$. Here, all the wave growth values are presented according to the above definition of β.

3. Growth rate calculations

3.1. *Logarithmic and 1/7-power-law profiles*

Conte & Miles (1959) gave accurate numerical computations of the wave growth rates for a logarithmic wind profile of the form

$$U(z) = U_1 \ln(z/z_0), \tag{3.1}$$

in which U_1 is as defined previously and the roughness height z_0 is determined empirically, the most frequently used expression given by Charnock (1955) as $z_0 = \alpha_{ch} u_*^2/g$ where $\alpha_{ch} \sim 0.011 - 0.018$ is Charnock's constant. Note that for the above logarithmic wind profile the critical height z_c, where the wind velocity equals the phase velocity c_0, is $z_c = z_0 \exp(c_0/U_1)$. The computational results of Conte & Miles (1959) for the dimensionless growth rate β were tabulated against c_0/U_1 for three values of the parameter $\Omega = gz_0/U_1^2$, which is directly related to Charnock's constant by $\Omega = \kappa^2 \alpha_{ch}$.

Besides the logarithmic profile the most commonly used wind profile is the so-called 1/7-power-law profile, which is defined as

$$U_p(z) = U_{10}(z/10)^{1/7}, \tag{3.2}$$

where U_{10} is the wind velocity at 10 m height. Since the comparisons are to be made with the results of the logarithmic wind profile in terms of the dimensionless parameters (c_0/U_1) and Ω, it is necessary to modify equation (3.2) for making it compatible with the logarithmic profile. The dimensionless critical height kz_c for the logarithmic wind profile is

$$kz_c = \Omega(c_0/U_1)^{-2} \exp(c_0/U_1). \tag{3.3}$$

For meaningful comparisons the dimensionless quantities (c_0/U_1) and Ω must be the same, which, in view of (3.3), requires kz_c be the same. Equation (3.2) is then modified as

$$U_p(z) = U_{1p}(z/z_0 - 1)^{1/7} \quad \text{where} \quad U_{1p} = c_0/[\exp(c_0/U_1) - 1]^{1/7}, \tag{3.4}$$

making $U_p(z_0) = 0$ and $z_c = z_0 \exp(c_0/U_1)$ as in (3.1). In this manner, the direct dependency of (3.4) on the dimensionless parameters (c_0/U_1) and Ω is ascertained.

In Table 1 the wave growth rates obtained for the logarithmic and 1/7-power-law-profiles are given for a range of c_0/U_1 values and three different Ω's. The differences reveal the effect of wind profile on the growth rates. These differences are remarkable in view of the slight variations between two profiles, as can be seen in Figure 1. On the other hand, the order of magnitude agreement between the results in a way supports the reliability of the shear flow approach as a means of estimating wave growth rates.

Figure 2 depicts the real and imaginary parts of $W(z)$ the $z-$dependent part of the disturbed vertical air velocity component $w(x, z, t)$. All the graphs are made for $c_0/U_1 = 5$ with three different Ω's; 3×10^{-3}, 1×10^{-2}, and 2×10^{-2} from top to bottom, respectively.

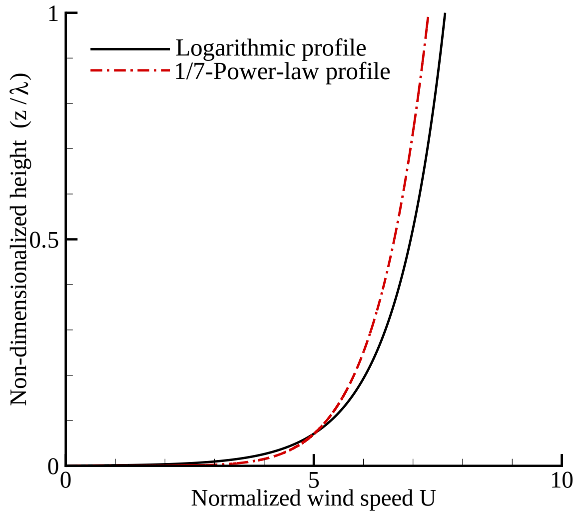

FIGURE 1. Comparison of logarithmic and 1/7-power-law wind profiles.

	$\Omega = 3 \times 10^{-3}$		$\Omega = 1 \times 10^{-2}$		$\Omega = 2 \times 10^{-2}$	
c_0/U_1	Logarithmic	1/7 Power	Logarithmic	1/7 Power	Logarithmic	1/7 Power
1	3.533	0.165	3.233	0.133	2.744	0.113
2	3.412	0.865	3.299	0.672	2.925	0.497
3	3.431	1.964	3.205	1.487	2.775	1.008
4	3.428	3.149	2.962	2.166	2.424	1.370
5	3.297	4.067	2.544	2.481	1.907	1.448
6	2.971	4.413	1.963	2.316	1.287	1.209
7	2.438	4.057	1.289	1.755	0.677	0.772
8	1.748	3.244	0.647	1.020	0.224	0.333
9	1.018	2.022	0.194	0.368	0.026	0.049
10	0.410	0.951	0.018	0.041	0.0002	0.0005

TABLE 1. Logarithmic and 1/7-power-law profile comparisons of the dimensionless growth rate β for a range of c_0/U_1 values and for three different Ω's.

The graphs on the left column show $W_R(z)$ and $W_I(z)$ for the logarithmic wind profile while those on the right for the 1/7-power-law wind profile. Recalling equations (2.4) and (2.7) the imaginary part of $W(z)$, $W_I(z)$ is more important as it determines the growth rate of waves. Since $\Im(I_c)$ is a weighted integration of $W_I(z)$ it is acceptable to conclude that the greater the area under the imaginary part the higher the growth rate. Then, considering for instance the graphs from top to bottom for both columns it is fairly easy to see that the growth rates would be getting less as we proceed from top to bottom. That is indeed the case as can be observed for row $c_0/U_1 = 5$ from Table 1.

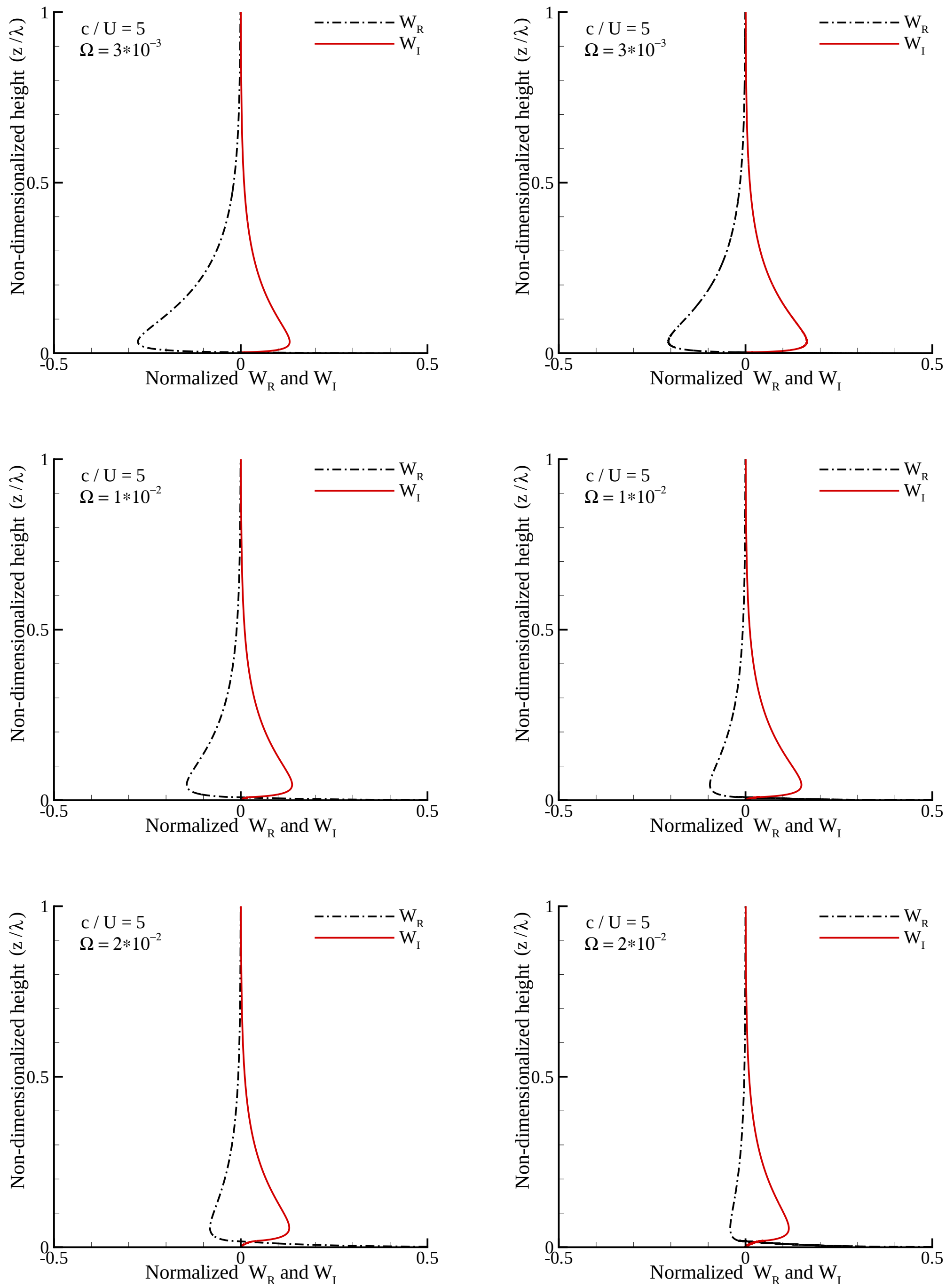

FIGURE 2. Change of real W_R and imaginary W_I components of normalized vertical air velocity with non-dimesionalized height for logarithmic wind profile (*Left column*) and 1/7-power-law profile (*Right column*).

4. Concluding remarks

The problem of wave generation by wind is considered from the point of view of shear instabilities as modelled by Rayleigh's instability equation. Employing the solution technique developed by Beji & Nadaoka (2004) for arbitrary wind profiles wave growth rate computations are performed for logarithmic and 1/7-power-law wind profiles. Numerical values show differences depending on the wind profile and clearly indicate the determining role of the individual wind profile on the wave growth rates. However, it must also be observed that these differences are moderate and maintain a meaningful level thus supporting the concept of shear instabilities as an acceptable tool in calculating growth rates. The vertical velocity $W(z)$ profiles provide a visual insight into the computations. In particular, the area under the curve of imaginary part $W_I(z)$ basically determines the extent of the growth rate.

Further studies may elaborate on more complicated wave forms other than the typically used simple sinusoidal form. For instance, a group of waves representing the random nature of waves or Stokes waves involving nonlinear aspects of the problem as in Sajjadi, Hunt & Drullion (2016) may be among the possible directions for future work.

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