INT. J. MATH. EDUC. SCI. TECHNOL., 1992, VOL. 23, NO. 2, 167–173

Investigations on cubic polynomials

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(*Received 16 August 1988*)

Several investigations on cubic polynomials are presented. The studies reported here deal with certain relationships which hold between the roots of the general cubic equation and other appropriate parameters derived from the same polynomial. Each section is organized in a monotonic sequence; that is, statement, demonstration and verification.

1. Introduction

Despite the fact that efforts made to solve cubic—and quartic—polynomials culminated in the well-known works of Galois about a hundred and fifty years ago [1], I studied the subject further to find out if there were points still beyond the current knowledge. This paper, being in many aspects suggestive rather than instructive, is a product of that attempt.

The following section contains an expression indicating a sort of 'balance' between the roots of the general cubic polynomial and those of its first derivative. Section 3 concerns itself with a different approach of obtaining the exact zeros. A method of reducing the order of polynomials for computing roots approximately is introduced in section 4. The paper concludes with two simple formulae yielding the roots of a cubic for the case when the equation has double roots.

2. A balance of roots

Statement. Consider a third-degree polynomial

$$P_3(x)=x^3+ax^2+bx+c$$

with its roots x_1, x_2 and x_3. Further, suppose that the derivative function

$$P_2(x)=3x^2+2ax+b$$

has the roots Θ_1, Θ_2 (see the figure). We then claim that the following relationship holds regardless of the choice of roots (i.e. x_j, Θ_α may be any one of the indicated roots: $j=1,2,3$ and $\alpha=1,2$).

$$\frac{x_2-\Theta_1}{x_1-\Theta_1}-\frac{x_2-\Theta_2}{x_3-\Theta_2}=\frac{(x_2-x_1)-(x_3-x_2)}{(x_1-x_3)}$$

Demonstration. To illustrate the above statement numerically, let

$$p_3(x)=x^3+x^2-4x+6=[x-(1+\mathrm{i})][x-(1-\mathrm{i})][x+3]$$

$$dp_3(x)/dx=p_2(x)=3x^2+2x-4=3[x+(1-\sqrt{13})/3][x+(1+\sqrt{13})/3]$$

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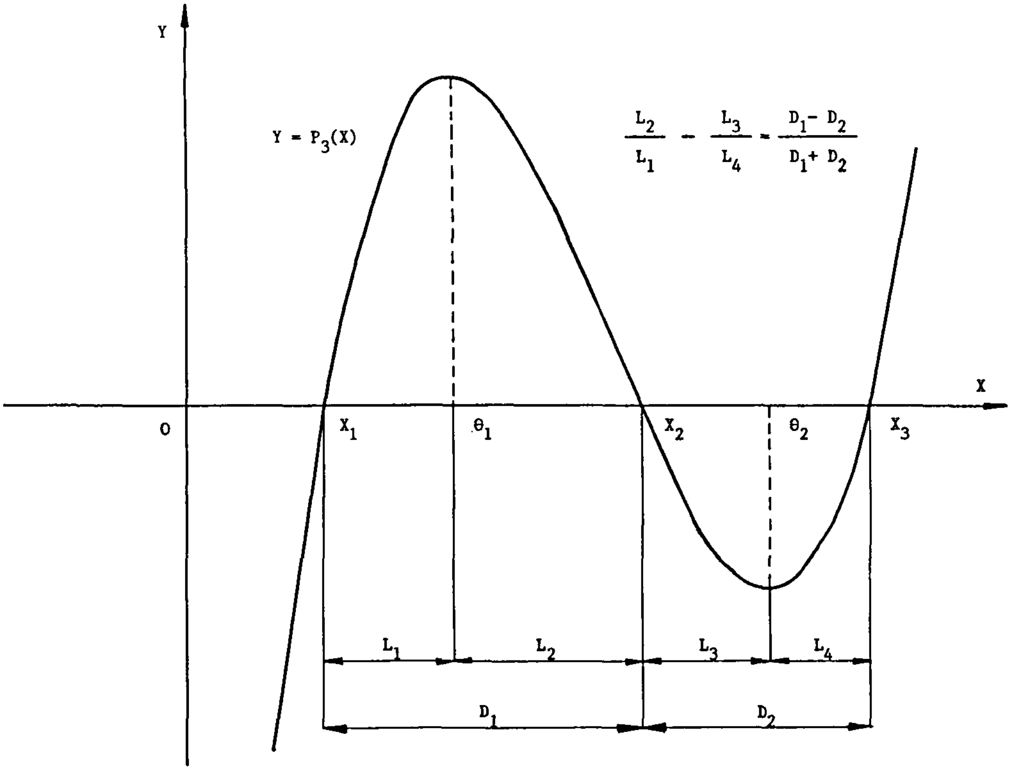

and if we denote $x_1=(1+i)$, $x_2=(1-i)$, $x_3=-3$, $\Theta_1=(-1+\sqrt{13})/3$, $\Theta_2=-(1+\sqrt{13})/3$ then,

$$\frac{(1-i)+(1-\sqrt{13})/3}{(1+i)+(1-\sqrt{13})/3}-\frac{(1-i)+(1+\sqrt{13})/3}{-3+(1+\sqrt{13})/3}=\frac{[(1-i)-(1+i)]-[-3-(1-i)]}{[(1+i)-(-3)]}$$

after some simple algebra we arrive at the result

$$\frac{13-16i}{17}=\frac{13-16i}{17}$$

It is worthwhile to note that interchanging Θ_1 with Θ_2 would not affect the result since the right-hand side of the equality is independent of Θ_1 and of Θ_2. Moreover, by assuming that all the roots are real and that $x_1<\Theta_1<x_2<\Theta_2<x_3$ as a special case, a geometric interpretation can be made. The figure depicts such a pattern.

Verification. A straightforward verification can be achieved through direct substitution. Let the zeros be x_1, x_2, x_3, then

$$P_3(X)=X^3-(x_1+x_2+x_3)X^2+(x_1x_2+x_1x_3+x_2x_3)X-x_1x_2x_3$$

$$dP_3(X)/dX=P_2(X)=3X^2-2(x_1+x_2+x_3)X+(x_1x_2+x_1x_3+x_2x_3)$$

and $\Theta_1=(\Sigma+\sqrt{\Delta})/3$, $\Theta_2=(\Sigma-\sqrt{\Delta})/3$ (the suffices may of course be interchanged) where

$$\Sigma=(x_1+x_2+x_3)$$
$$\Delta=(x_1+x_2+x_3)^2-3(x_1x_2+x_1x_3+x_2x_3)$$

Starting with the left-hand side of the equality

$$\frac{x_2-(\Sigma+\sqrt{\Delta})/3}{x_1-(\Sigma+\sqrt{\Delta})/3}-\frac{x_2-(\Sigma-\sqrt{\Delta})/3}{x_3-(\Sigma-\sqrt{\Delta})/3}$$

and performing some algebra with the substitution of Σ, Δ yield

$$\frac{(x_2-x_1)-(x_3-x_2)}{(x_1-x_3)}$$

which is the same as the right-hand side.

3. Exact solutions: a different approach

Statement. Inspired by the previous statement I further proceed to show a direct connection between the roots of a cubic equation and the quantities affected by differentiating the polynomial itself. Consider

$$P_3(x)=x^3+ax^2+bx+c=0 \qquad x_1, x_2, x_3$$
$$dP_3(x)/dx=P_2(x)=3x^2+2ax+b=0 \qquad \Theta_1, \Theta_2$$
$$d^2P_3(x)/dx^2=P_1(x)=6x+2a=0 \qquad \Theta$$

$\Theta=-a/3$ is simply the root of the function obtained through twice differentiating the general cubic polynomial. (This quantity is the same as the constant employed in the Tschirnhaus transformation [1].) If the solutions to the roots x_1, x_2, x_3 are of the following forms [1, 2]:

$$x_1=A+\alpha_{11}\ \sqrt[3]{(B+\alpha_1\sqrt{C})}+a_{12}\ \sqrt[3]{(B+\alpha_2\sqrt{C})}$$
$$x_2=A+\alpha_{21}\ \sqrt[3]{(B+\alpha_1\sqrt{C})}+\alpha_{22}\ \sqrt[3]{(B+\alpha_2\sqrt{C})}$$
$$x_3=A+\alpha_{31}\ \sqrt[3]{(B+\alpha_1\sqrt{C})}+\alpha_{32}\ \sqrt[3]{(B+\alpha_2\sqrt{C})}$$

where $\alpha_1=+1$, $\alpha_2=-1$ are the roots of $\alpha^2=+1$, and $\alpha_{11}=\alpha_{12}=+1$, $\alpha_{21}=(-1+\sqrt{3}i)/2$, $\alpha_{22}=-(1+\sqrt{3}i)/2$, $\alpha_{31}=\alpha_{22}$, $\alpha_{32}=\alpha_{21}$ are the roots of $\alpha^3=+1$, then the foregoing parameters can be expressed as follows:

$$A=\Theta=-a/3$$
$$B=-\tfrac{1}{2}P_3(\Theta)$$
$$C=[-\tfrac{1}{2}P_3(\Theta)]^2+[\tfrac{1}{3}P_2(\Theta)]^3$$

Demonstration. As a numerical example, let

$$p_3(x)=x^3+x^2+x+1$$
$$p_2(x)=3x^2+2x+1$$
$$p_1(x)=6x+2 \qquad \Theta=-\tfrac{1}{3}$$

The other quantities would read

$$p_3(\Theta)=(-\tfrac{1}{3})^3+(-\tfrac{1}{3})^2+(-\tfrac{1}{3})+1=+20/27$$

$$p_2(\Theta)=3(-\tfrac{1}{3})^2+2(-\tfrac{1}{3})+1=+\tfrac{2}{3}$$

$$A=-\tfrac{1}{3}$$

$$B=-\tfrac{1}{2}(+20/27)=-10/27$$

$$C=[-\tfrac{1}{2}(+20/27)]^2+[\tfrac{1}{3}(+\tfrac{2}{3})]^3=+108/729$$

$$x_1=-\tfrac{1}{3}+\sqrt[3]{[(-10/27)+\sqrt{(108/729)}]}+\sqrt[3]{[(-10/27)-\sqrt{(108/729)}]}=-1$$

Similarly, $x_2=+\mathrm{i}$, and $x_3=-\mathrm{i}$.

Verification. A simple substitution gives

$$B=-\tfrac{1}{2}P_3(\Theta)=-\tfrac{1}{2}[(-a/3)^3+a(-a/3)^2+b(-a/3)+c]$$
$$=(1/27)[-a^3+(9/2)ab-(27/2)c]$$
$$P_2(\Theta)=3(-a/3)^2+2a(-a/3)+b=\tfrac{1}{3}[-a^2+3b]$$
$$C=[-\tfrac{1}{2}P_3(\Theta)]^2+[\tfrac{1}{3}P_2(\Theta)]^3$$
$$=[(-1/54)(2a^3-9ab+27c)]^2+[(1/9)(-a^2+3b)]^3$$
$$=(1/18)^2[-3a^2b^2+12b^3+12a^3c+81c^2-54abc]$$

which are in complete agreement with the known solutions [2]. The interested reader is directed to reference [3] where a full account of the subject with historical remarks and various solution techniques can be found.

Before closing this argument let us briefly look at the case of quadratic polynomials. Consider

$$P_2(x)=x^2+ax+b \qquad x_1,x_2$$
$$dP_2(x)/dx=P_1(x)=2x+a \qquad \Theta=-a/2$$
$$P_2(\Theta)=\tfrac{1}{4}(4b-a^2)$$

If we assume the roots x_1, x_2 as

$$x_1=A+\alpha_1\sqrt{B} \qquad \text{and} \qquad x_2=A+\alpha_2\sqrt{B}$$

it is possible to show that

$$A=\Theta=-a/2 \qquad B=[-P_2(\Theta)/1]$$

and $\alpha_1=+1$, $\alpha_2=-1$ are the roots of $\alpha^2=+1$. Essentially the above simple deduction has established a basis for extending the application to cubic polynomials.

4. An approximate method

Statement. We now take up the problem of obtaining approximate solutions and claim that a degree-three equation can be degenerated into a quadratic in order to get approximate roots. If we are given,

$$P_3(x)=x^3+ax^2+bx+c \qquad x_1,x_2,x_3$$

the following quadratic equations render at least one of the three roots, the solution accuracy being increased with ascending order.

Order neglected	*Corresponding equation*
3rd	$ax^2+bx+c=0$
4th	$(b-a^2)x^2+(c-ab)x-ac=0$
5th	$[a(a^2-2b)+c]x^2+[b(a^2-b)-ac]x+c(a^2-b)=0$

Higher order approximations can be computed in a straightforward manner, but the equations become cumbersome and therefore impractical.

Demonstration. To observe the nature of the proposed method, four particularly selected equations are inspected. For the sake of brevity, only fourth and fifth order approximations are considered. The approximate equations are divided by the coefficients of x^2 in order to avoid large numbers.

Real roots

$p_3(x)=(x-0{\cdot}7)(x+2{\cdot}5)(x+4{\cdot}0)=x^3+5{\cdot}80x^2+5{\cdot}45x-7{\cdot}00$

4th order: $x^2+1{\cdot}37x-1{\cdot}44$	$x_1=+0{\cdot}70$	$x_2=-2{\cdot}07$
5th order: $x^2+1{\cdot}56x-1{\cdot}58$	$x_1=+0{\cdot}70$	$x_2=-2{\cdot}26$

Real roots, larger magnitude

$p_3(x)=(x+7{\cdot}0)(x+9{\cdot}0)(x+15{\cdot}0)=x^3+31{\cdot}0x^2+303{\cdot}0x+945{\cdot}0$

4th order: $x^2+12{\cdot}84x+44{\cdot}52$	$x_1=-6{\cdot}42+1{\cdot}82i$	$x_2=-6{\cdot}42-1{\cdot}82i$
5th order: $x^2+14{\cdot}23x+52{\cdot}03$	$x_1=-7{\cdot}12+1{\cdot}19i$	$x_2=-7{\cdot}12-1{\cdot}19i$
10th order: $x^2+15{\cdot}87x+62{\cdot}09$	$x_1=-7{\cdot}00$	$x_2=-8{\cdot}87$

Complex conjugate roots, magnitude less than third root

$p_3(x)=[x+(0{\cdot}5+0{\cdot}5i)][x+(0{\cdot}5-0{\cdot}5i)][x+2{\cdot}0]=x^3+3{\cdot}0x^2+2{\cdot}5x+1{\cdot}0$

4th order: $x^2+1{\cdot}00x+0{\cdot}46$	$x_1=-(0{\cdot}50+0{\cdot}46i)$	$x_2=-(0{\cdot}50-0{\cdot}46i)$
5th order: $x^2+1{\cdot}02x+0{\cdot}50$	$x_1=-(0{\cdot}51+0{\cdot}49i)$	$x_2=-(0{\cdot}51-0{\cdot}49i)$

Complex conjugate roots, magnitude greater than third root

$p_3(x)=[x-(3{\cdot}0+2{\cdot}0i)][x-(3{\cdot}0-2{\cdot}0i)][x-0{\cdot}5]=x^3-6{\cdot}5x^2+16{\cdot}0x-6{\cdot}5$

4th order: $x^2-3{\cdot}71x+1{\cdot}61$	$x_1=+0{\cdot}50$	$x_2=+3{\cdot}21(?)$
5th order: $x^2-5{\cdot}17x+2{\cdot}33$	$x_1=+0{\cdot}50$	$x_2=+4{\cdot}67(?)$

Convergence is obviously faster for the roots closer to unity. Otherwise, a computer routine is necessary as disclosed in the second example. It can also be demonstrated that convergence speed is greatly dependent on the sign of the roots. Unlike most of the other approximate solution techniques, this method does not require any initial guess and the iteration proceeds towards smaller roots from the very beginning. The method however fails to give any solution and convergence becomes unstable when roots are equal or considerably closer in magnitude. The reason will be clear in the next section.

Verification. The method is now explained to justify the preceding results. Let us start with

$$P_3(x)=x^3+ax^2+bx+c=0 \qquad x_1, x_2, x_3$$

and introduce a transformation such that $x=kz$ where k is an unknown constant to be determined later. Upon substitution,

$$k^3z^3+ak^2z^2+bkz+c=0$$

$$z^3+(a/k)z^2+(b/k^2)z+(c/k^3)=0$$

We have tacitly assumed that $k\neq 0$. To determine k, we set $(c/k^3)=1$, namely $k=\sqrt[3]{c}$, and obtain

$$z^3+(a/\sqrt[3]{c})z^2+(b/(\sqrt[3]{c})^2)z+1=0$$

Due to the fact that $z_1z_2z_3=-1$ the above equation has at least one root whose magnitude is less than unity, unless all the roots are equal (the deficiency mentioned earlier originates from this exception). Leaving this particular case aside, we can proceed as follows. If

$$|z_i|<1 \qquad \text{then} \qquad |z_i|^3<|z_i|^2<|z_i|$$

Here z_i denotes the root(s) less than unity. A rough approximation can readily be made by setting $z^3=0$:

$$(a/\sqrt[3]{c})z^2+(b/(\sqrt[3]{c})^2)z+1=0$$

Recalling $z=x/k$, $k=\sqrt[3]{c}$,

$$ax^2+bx+c=0$$

Let us refine the approximation. Consider the transformed equation

$$z^3=-[(a/\sqrt[3]{c})z^2+(b/(\sqrt[3]{c})^2)z+1]$$

Multiplying both sides by z gives

$$z^4=-[(a/\sqrt[3]{c})z^3+(b/(\sqrt[3]{c})^2)z^2+z]$$

We neglect z^4 and make use of the previous equation for z^3

$$-(a/\sqrt[3]{c})[(a/\sqrt[3]{c})z^2+(b/(\sqrt[3]{c})^2)z+1]+(b/(\sqrt[3]{c})^2)z^2+z=0$$

After employing $z=x/\sqrt[3]{c}$, the fourth order term being neglected, we certainly have a better approximation:

$$(b-a^2)x^2+(c-ab)x-ac=0$$

The procedure can similarly be repeated to compute higher order approximations. It is worthy of remark that the approximate equations can be obtained without employing the transformation $x=kz$, that is to say, multiplying the equation by x, neglecting the highest order term and replacing x^3 by $-(ax^2+bx+c)$.

5. Case of double roots

Statement. The special case of double roots is now analysed. If the equation

$$P_3(x)=x^3+ax^2+bx+c=0 \qquad x_1, x_2, x_3$$

has double roots, $x_1 = x_2$, besides the third root x_3, the roots may be expressed as

$$x_1 = x_2 = \frac{(ab-9c)}{2(3b-a^2)} \qquad x^3 = \frac{(a^3-4ab+9c)}{(3b-a^2)}$$

Demonstration. Consider

$$p_3(x) = (x-2)^2(x+3) = x^3 - x^2 - 8x + 12$$

$$x_1 = x_2 = \frac{[(-1)(-8)-9(+12)]}{2[3(-8)-(-1)^2]} = +\frac{100}{50} = +2$$

$$x_3 = \frac{[(-1)^3-4(-1)(-8)+9(+12)]}{3(-8)-(-1)^2} = -\frac{75}{25} = -3$$

Verification. Denote $x_1 = x_2 = x$, and $x_3 = y$. We then have (see the first equation of the verification in section 2)

$$2x + y = -a$$

$$x^2 + 2xy = +b$$

$$x^2 y = -c$$

By using the above set, one can easily show that

$$x^3 - 2x^2 y + xy^2 = \tfrac{1}{2}(9c - ab)$$

$$x(x-y)^2 = \tfrac{1}{2}(9c - ab)$$

On the other hand,

$$(x-y)^2 = a^2 - 3b$$

thus,

$$x = \frac{(ab-9c)}{2(3b-a^2)}$$

The third root is obviously

$$y = -a - 2x = -a - (ab-9c)/(3b-a^2)$$

$$y = \frac{(a^3-4ab+9c)}{(3b-a^2)}$$

References

[1] Stewart, I., 1973, *Galois Theory* (New York: Halstead), pp. 161–163.

[2] Borofsky, S., 1950, *Elementary Theory of Equations* (New York: Macmillan), pp. 115–130.

[3] Aleksandrov, A. D., Kolmogorov, A. N., and Lavrent'ev, M. A., 1963, *Mathematics: Its Content, Methods, and Meaning*, Vol. 1 (Cambridge, Mass: MIT Press), pp. 261–310, 190–193.