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Kadomtsev–Petviashvili type equation for entire range of relative water depths

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ABSTRACT

A Kadomtsev–Petviashvili type equation valid for the entire range of relative water depths; namely, shallow, intermediate, and deep, is derived. The new equation is capable of simulating the shallow water cnoidal waves and the deep water second-order Stokes waves equally well besides accounting for wave shoaling due to varying water depths. Linear shoaling properties of the equation for unidirectional sinusoidal waves are in complete agreement with the energy flux concept. A finite-difference scheme is adopted for numerical solution of the equation to demonstrate its performance against test cases and its wide range of possible applications.

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1. Introduction

Since its first derivation by Kadomtsev and Petviashvili (1970) for the purpose of studying the stability of solitary waves the Kadomtsev and Petviashvili equation or shortly the KP equation has gained an appreciable interest, particularly in the field of non-linear dispersive water wave modeling. The KP equation is viewed as the weakly directional version of the Korteweg and deVries or KdV equation (Korteweg and deVries, 1895), which in turn is a one-way propagation form of one-dimensional Boussinesq (1872) equation: a weakly non-linear weakly dispersive wave model.

Bryant (1982) derived a set of non-linear evolution equations which, depending on the specification of coefficients, could describe fully dispersive waves as well as weakly dispersive waves of the KP type. Using these equations, Bryant (1982) studied obliquely intersecting permanent waves and indicated the rather good performance of the KP equation for the cases considered.

Tsuji and Oikawa (2007) considered the oblique interaction of two solitons of the same amplitude by employing an extended KP equation with cubic non-linearity. Kodama (2010) gave an extensive survey of solitons and their modeling by KP equation. Yeh, Li, and Kodama (2010) investigated analytically and experimentally Mach reflection of an obliquely incident solitary wave while Li, Yeh, and Kodama (2011) made an in-depth review of the same subject.

In laboratory experiments, Hammack, Scheffner, and Segur (1989) generated finite-amplitude two-dimensional shallow water waves and mathematically described them by the exact solutions of the KP equation. The capabilities of the KP equation are noted to be not restricted to weakly directional waves; the extend of directionality encompassed by the KP

equation is observed to be beyond the usual validity range of it but only slightly hindered by its linear non-dispersive character in transverse direction. Such satisfactory performance of the weakly non-linear models has quite recently been pointed out by Stiassnie (2017) with reference to the experimental confirmations of Bonnefoy et al. (2016). From this point of view, for practical application purposes it becomes much more important to place emphasis on the dispersion and shoaling characteristics of the equation rather than higher non-linearity. The work presented here aims precisely at developing such an equation that possesses excellent linear dispersion and shoaling characteristics with just sufficient non-linearity.

Nadaoka, Beji, and Nakagawa (1994) introduced a vertically integrated non-linear wave model without restriction on relative depth. Likewise, making use of the variational principle of Luke (1967), Isobe (1994) gave various wave models for different depth-dependency functions. Continuing in the same line, Nadaoka, Beji, and Nakagawa (1997) presented a weakly non-linear wave model with full dispersion properties by employing the Galerkin method which provided a set of coupled momentum equations for the components of horizontal velocity vector. Together with the continuity equation the wave model results in $2N + 1$ equations, N being the number of wave numbers associated with velocity components contributing to the bandwidth of the wave field. Taking only three components $N = 3$ is quite sufficient for simulating a broad-banded wave spectrum while taking just a single component $N = 1$ is good enough for the simulation of waves with a narrow-banded spectrum. The special case of a single component was further elaborated by Beji and Nadaoka (1997) to produce a non-linear mild-slope

equation. Here, based on the work of Beji and Nadaoka (1997) a KP type wave equation is produced for modeling narrow-banded weakly non-linear water waves over varying bathymetry for arbitrary relative depths.

The newly derived equation is discretized by a Crank–Nicolson type finite-difference formulation and various test cases are carried out. First, shoaling properties of the equation for sinusoidally changing bathymetry are explored for incident waves of two different relative depths. Then, obliquely intersecting two-dimensional cnoidal waves and second-order Stokes waves are simulated. Finally, Whalin's (1971) experimental measurements of non-linear waves over a converging zone are compared with numerical results. The simulations clearly demonstrate that the KP type equation derived here is capable of simulating propagation of linear and non-linear waves over the entire range of relative water depths with acceptable accuracy.

2. One component wave model in combined form

Before proceeding to the derivation of the KP type equation, the combined form of the one component wave model of Nadaoka, Beji, and Nakagawa (1997) is recaptured. For the special case of a single component, Nadaoka, Beji, and Nakagawa (1997) give the following continuity and momentum equations.

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot \left[\left(\frac{C_p^2}{g} + \zeta \right) \mathbf{u}_0 \right] = 0, \quad (1)$$

$$\begin{aligned} & C_p C_g \frac{\partial \mathbf{u}_0}{\partial t} + C_p^2 \nabla \left[g \zeta + \zeta \frac{\partial w_0}{\partial t} + \frac{1}{2} (\mathbf{u}_0 \cdot \mathbf{u}_0 + w_0^2) \right] \\ &= \frac{\partial}{\partial t} \left\{ \frac{C_p(C_p - C_g)}{k^2} \nabla (\nabla \cdot \mathbf{u}_0) + \nabla \left[\frac{C_p(C_p - C_g)}{k^2} \right] (\nabla \cdot \mathbf{u}_0) \right\} \end{aligned} \quad (2)$$

where ζ is the free surface displacement as measured from the still water level, \mathbf{u}_0 is the horizontal velocity vector with components (u_0, v_0) , and w_0 the vertical velocity component all at the still water level $z = 0$. C_p , C_g , and k denote respectively the phase and group velocities and wave number, computed according to the linear theory dispersion relationship $\omega^2 = gk \tanh kh$ for a prescribed dominant frequency ω and a local depth $h = h(x, y)$. g is the gravitational acceleration and ∇ stands for two-dimensional horizontal gradient operator with components $(\partial/\partial x, \partial/\partial y)$.

The above equations constitute a wave model for narrow-banded weakly non-linear waves propagating over varying depths. In the non-dispersive limit when $C_p \simeq C_g \simeq C_s = (gh)^{1/2}$ the equations reduce to Airy's shallow water equations. If C_p and C_g are approximated to the second-order as $C_s(1 - k^2 h^2/6)$ and

$C_s(1 - k^2 h^2/2)$ respectively and used in the dispersive terms yields the Boussinesq equations for varying depth as given by Peregrine (1967).

By cross-differentiations and use of zeroth-order relations for non-linear terms Equations (1) and (2) may be combined into a single non-linear wave equation for ζ as detailed in Beji and Nadaoka (1997).

$$\begin{aligned} \zeta_{tt} - \frac{C_p^2}{r} \nabla^2 \zeta - \frac{C_p^2(1-r)}{r\omega^2} \nabla^2 \zeta_{tt} - \frac{3g}{2r} \left(3 - 2r - \frac{\omega^2 C_p^2}{g^2} \right) \nabla^2 (\zeta^2) \\ - \nabla \left(\frac{C_p^2}{r} \right) \cdot \nabla \zeta - \frac{1}{\omega^2} \left[\nabla \left((1-r) \frac{C_p^2}{r} \right) - \frac{C_p^2}{r} \nabla r \right] \cdot \nabla \zeta_{tt} = 0 \end{aligned} \quad (3)$$

where $r = C_g/C_p = 1/2(1 + 2kh/\sinh 2kh)$ and ∇ stands for two-dimensional horizontal gradient operator as before while subscript t denotes partial differentiation with respect to time. Equation (3) differs slightly from the equation given in Beji and Nadaoka (1997) with respect to the last two terms. If the zeroth-order relation for harmonic motion $\zeta_{tt} = -\omega^2 \zeta$ is used in the last term and combined with the preceding linear shoaling term the result becomes identical with that of Beji and Nadaoka (1997). Further, removal of the harmonic time-dependency from the relevant terms and neglect of the non-linear term leads, after some manipulations, to Berkhoff's (1972) mild-slope equation, which in turn, for constant depth gives the Helmholtz equation and Lamb's (1932) shallow water equation (p. 283) as special cases.

3. New KP equation and its dispersion relationship

A KP type equation based on the combined one-component model of Beji and Nadaoka (1997) as expressed in Equation (3) is now derived. The classical approach proceeds by employing a non-dimensional parameter for scaling the y-co-ordinate. Implicitly, the parameter is assumed proportional to the ratio of transverse wave number to the main propagation wave number K_y/K_x . Then, except for the lowest-order term all the transverse terms are dropped. In this work, the discharge of these definite transverse terms is carried out in an informal manner by pointing out their physical functions. Namely, the dispersive and non-linear terms containing the y – derivative are all dropped as it is done by following purely mathematical arguments.

3.1. KP type equation for arbitrary relative water depths

For deriving a KP type equation from Equation (3) the x – direction is taken as the main propagation direction with all the relevant terms included while in the y – direction only the term representing non-dispersive linear propagation is retained. All the

other y -dependent terms are dropped. Such a truncation renders the propagation in the y -direction linear and non-dispersive; however, it does not imply a strictly *weak-directionality* as commonly termed in the literature. The weakness in directionality may be associated with the asymmetric propagation properties of the final wave equation after the equation is cast into a one-way propagation model via co-ordinate transformation. The reduced form of Equation (3) then reads

$$\begin{aligned} \zeta_{tt} - \frac{C_p^2}{r}(\zeta_{xx} + \zeta_{yy}) - \frac{C_p^2(1-r)}{r\omega^2}\zeta_{xxtt} - \frac{3g}{2r}\left(3 - 2r - \frac{\omega^2 C_p^2}{g^2}\right)(\zeta^2)_{xx} \\ - \frac{C_p}{r^2}(2rC_{px} - C_p r_x)\zeta_x - \frac{C_p}{r^2\omega^2}[2r(1-r)C_{px} \\ - (1+r)C_p r_x]\zeta_{xtt} = 0 \end{aligned} \quad (4)$$

where C_{px} and r_x denote respectively the x -derivatives of C_p and $r = C_g/C_p$. A co-ordinate system moving in the positive x -direction with the phase velocity C_p is introduced so that the evolutions of the wave form in this moving system are slow, permitting to write the following new co-ordinates:

$$\sigma = x - C_p t, \quad \tau = \varepsilon t, \quad (5)$$

where ε is a small parameter indicating the weak changes of the wave form in time in the moving co-ordinate system. Expressing the terms in Equation (4) in the new co-ordinate system gives

$$\begin{aligned} \zeta_{tt} &= C_p^2 \zeta_{\sigma\sigma} - 2\varepsilon C_p \zeta_{\sigma\tau} + \varepsilon C_p C_{p\sigma} \zeta_{\sigma}, \\ \zeta_{xtt} &= C_p^2 \zeta_{\sigma\sigma\sigma} - 2\varepsilon C_p \zeta_{\sigma\sigma\tau} + \varepsilon C_p C_{p\sigma} \zeta_{\sigma\sigma}, \quad \zeta_{xx} = \zeta_{\sigma\sigma}, \\ \zeta_{xxtt} &= C_p^2 \zeta_{\sigma\sigma\sigma\sigma} - 2\varepsilon C_p \zeta_{\sigma\sigma\sigma\tau} + \varepsilon C_p C_{p\sigma} \zeta_{\sigma\sigma\sigma} \end{aligned} \quad (6)$$

where the terms containing the spatial derivative of C_p have also been labeled by ε to indicate they are an order higher, and the terms proportional to ε^2 are all neglected. Substituting the expressions in Equation (6) into Equation (4) and re-arranging results in

$$\begin{aligned} -2\varepsilon C_p \zeta_{\sigma\tau} - \frac{C_p^2(1-r)}{r} \zeta_{\sigma\sigma} + 2\varepsilon \frac{C_p^2(1-r)}{r\omega^2} \zeta_{\sigma\sigma\sigma\tau} \\ - \frac{C_p^4(1-r)}{r\omega^2} \zeta_{\sigma\sigma\sigma\sigma} - a(\zeta^2)_{\sigma\sigma} + \varepsilon(C_p C_{p\sigma} - \beta)\zeta_{\sigma} \\ - \varepsilon C_p^2 \left(\frac{C_p(1-r)C_{p\sigma}}{r\omega^2} + \gamma \right) \zeta_{\sigma\sigma\sigma} + 2\varepsilon \gamma C_p \zeta_{\sigma\sigma\tau} = \frac{C_p^2}{r} \zeta_{yy}, \end{aligned} \quad (7)$$

where the coefficients $a = 3g(3 - 2r - \omega^2 C_p^2/g^2)/2r$, $\beta = C_p(2rC_{px} - C_p r_x)/r^2$, and $\gamma = C_p[2r(1-r)C_{px} - (1+r)C_p r_x]/r^2\omega^2$ have been introduced for the ease of notation.

The next step is to put Equation (7) into a form readily integrable with respect to σ . Therefore, noting that both C_p and r are spatially varying quantities, the

following equalities correct to the second spatial derivatives of C_p and r may be written

$$\begin{aligned} \frac{C_p^2(1-r)}{r} \zeta_{\sigma\sigma} &= \left[\frac{C_p^2(1-r)}{r} \zeta_{\sigma} + \left(2C_p C_{p\sigma} - \frac{C_p(2rC_{p\sigma} - C_p r_x)}{r^2} \right) \zeta \right]_{\sigma} \\ \frac{C_p^3(1-r)}{r\omega^2} \zeta_{\sigma\sigma\sigma\tau} &= \left[\frac{C_p^3(1-r)}{r\omega^2} \zeta_{\sigma\sigma\tau} - \left(\frac{3C_p^2(1-r)rC_{p\sigma} - C_p^3 r_x}{r^2\omega^2} \right) \zeta_{\sigma\tau} \right]_{\sigma} \\ \frac{C_p^4(1-r)}{r\omega^2} \zeta_{\sigma\sigma\sigma\sigma} &= \left[\frac{C_p^4(1-r)}{r\omega^2} \zeta_{\sigma\sigma\sigma} - \left(\frac{4C_p^3(1-r)rC_{p\sigma} - C_p^4 r_x}{r^2\omega^2} \right) \zeta_{\sigma\sigma} \right]_{\sigma} \end{aligned} \quad (8)$$

Making use of Equation (8) in Equation (7), re-arranging and dividing the resulting equation by $-2C_p$ give

$$\begin{aligned} \frac{\partial}{\partial \sigma} [\varepsilon \zeta_{\tau} + \frac{C_p(1-r)}{2r} \zeta_{\sigma} - \varepsilon \frac{C_p^2(1-r)}{r\omega^2} \zeta_{\sigma\sigma\tau} \\ + \frac{C_p^3(1-r)}{2r\omega^2} \zeta_{\sigma\sigma\sigma} + \frac{a}{2C_p} (\zeta^2)_{\sigma} + \varepsilon \frac{C_{p\sigma}}{2} \zeta \\ + \varepsilon \frac{C_p}{r\omega^2} [(1-r)C_{p\sigma} + C_p r_x] \zeta_{\sigma\tau} \\ - \varepsilon \frac{C_p^2}{2r\omega^2} [(1-r)C_{p\sigma} + C_p r_x] \zeta_{\sigma\sigma}] + \frac{C_p}{2r} \zeta_{yy} = 0 \end{aligned} \quad (9)$$

The inverse transformation is carried out by using the following expressions.

$$\varepsilon \zeta_{\tau} = \zeta_t + C_p \zeta_x, \quad \varepsilon \zeta_{\sigma\tau} = \zeta_{xt} + C_p \zeta_{xx}, \quad \zeta_{\sigma} = \zeta_x, \text{ etc.} \quad (10)$$

Thus, the new KP type equation reads

$$\begin{aligned} \frac{\partial}{\partial x} [\zeta_t + \frac{C_p(1+r)}{2r} \zeta_x - \frac{C_p^2(1-r)}{r\omega^2} \zeta_{xtt} - \frac{C_p^3(1-r)}{2r\omega^2} \zeta_{xxx} \\ + \frac{a}{2C_p} (\zeta^2)_x + \frac{C_{px}}{2} \zeta + \frac{C_p}{r\omega^2} [(1-r)C_{px} + C_p r_x] \zeta_{xt} \\ + \frac{C_p^2}{2r\omega^2} [(1-r)C_{px} + C_p r_x] \zeta_{xx}] + \frac{C_p}{2r} \zeta_{yy} = 0 \end{aligned} \quad (11)$$

where the equation inside the square brackets corresponds to the KdV type equation given by Beji and Nadaoka (1997). The only difference is in the linear shoaling terms, which may all be combined into a single term by invoking the zeroth-order relations $\zeta_{xx} = -k^2 \zeta$ and $\zeta_{xt} = k\omega \zeta$, thus making it identical with the term in Beji and Nadaoka (1997). Expanded form of the new KP type equation is obtained by performing the x -differentiation,

$$\begin{aligned} \zeta_{xt} + \frac{C_p(1+r)}{2r} \zeta_{xx} - \frac{C_p^2(1-r)}{r\omega^2} \zeta_{xxtt} - \frac{C_p^3(1-r)}{2r\omega^2} \zeta_{xxxx} \\ + \frac{a}{2C_p} (\zeta^2)_{xx} + \frac{1}{2r^2} [r(1+2r)C_{px} - C_p r_x] \zeta_x \\ - \frac{C_p}{r^2\omega^2} [r(1-r)C_{px} - (1+r)C_p r_x] \zeta_{xtt} \\ - \frac{C_p^2}{2r^2\omega^2} [2r(1-r)C_{px} - (1+r)C_p r_x] \zeta_{xxx} \\ + \frac{C_p}{2r} \zeta_{yy} = 0 \end{aligned} \quad (12)$$

where $r = C_g/C_p$ and $a = 3g(3 - 2r - \omega^2 C_p^2/g^2)/2r$ as defined before. The above KP type equation

represents all the known KP-like equations as special cases: for constant depth when $C_{px} = 0$ and $r_x = 0$, setting $C_p \simeq C_s = (gh)^{1/2}$ and $r = 1$ in the coefficients of ζ_{xx} , ζ_{yy} , $(\zeta^2)_{xx}$, and $r = C_g/C_p \simeq [C_s(1 - k^2h^2/2)]/[C_s(1 - k^2h^2/6)] \simeq 1 - k^2h^2/3 + O(k^4h^4)$ in the coefficients of the dispersion terms ζ_{xxx} and ζ_{xxxx} as approximated by weakly dispersive theory results in

$$\zeta_{xt} + C_s \zeta_{xx} - \frac{h^2}{3} \zeta_{xxx} - C_s \frac{h^2}{6} \zeta_{xxxx} + \frac{3C_s}{4h} (\zeta^2)_{xx} + \frac{C_s}{2} \zeta_{yy} = 0, \quad (13)$$

where $C_s = \sqrt{gh}$ is the shallow water celerity. Equation (13) has mixed dispersion terms but may be easily put into the classical KP equation by using the zeroth-order relationship $\zeta_t = -C_s \zeta_x$ in the third term so that $-(h^2/3)\zeta_{xxx} = C_s(h^2/3)\zeta_{xxxx}$ and the final dispersion term becomes $C_s(h^2/3)\zeta_{xxxx} - C_s(h^2/6)\zeta_{xxxx} = C_s(h^2/6)\zeta_{xxxx}$ as expected. On the other hand, replacing the fourth term ζ_{xxxx} by $-\zeta_{xxx}/C_s$ gives $-(h^2/3)\zeta_{xxx} + (h^2/6)\zeta_{xxx} = -(h^2/6)\zeta_{xxx}$, which corresponds to the dispersion term in the KP-like equation based on the so-called BBM model (Benjamin, Bona, and Mahony, 1972). The terms proportional to C_{px} and r_x provide the wave equation with linear shoaling properties which are exact for sinusoidal waves having the same frequency specified for the wave model. This point is demonstrated in Section 5.1.

3.2. Dispersion relationship of KP equation

Dispersion relationship of the newly developed KP type equation is now obtained. Let $\zeta = \zeta_0 \exp[i(K_x x + K_y y - \Omega t)]$ represent a sinusoidal incident wave moving in an arbitrary direction on the xy plane, where the horizontal two-dimensional wave propagation takes place. Here, ζ_0 is the constant wave amplitude, i the imaginary unit, Ω the circular incident wave frequency, $K_x = K \cos \theta$ and $K_y = K \sin \theta$ the wave number components in the x - and y -directions, respectively. $\theta = \arctan(K_y/K_x)$ is the angle wave propagation direction makes with the x -axis. Substituting ζ into the linearized, constant depth form of Equation (12) gives

$$\frac{\Omega}{K_x} = \frac{C_p}{2} \left(\frac{(1+r) + C_p^2(1-r)(K_x/\omega)^2 + (K_y/K_x)^2}{r + C_p^2(1-r)(K_x/\omega)^2} \right), \quad (14)$$

where, as indicated before, C_p and $r = C_g/C_p$ are computed according to linear theory for a specified frequency ω and local depth $h = h(x, y)$ using $\omega^2 = gk \tanh kh$.

Denoting \mathbf{C}_{KP} as the phase velocity vector of the KP type equation and recalling the relationship between the frequency and celerity $\Omega = \mathbf{K} \cdot \mathbf{C}_{KP}$ with $\mathbf{K} = K_x \mathbf{i} +$

$K_y \mathbf{j}$ for directional waves, the dispersion relationship given by Equation (14) may be cast into the following form for the phase velocity vector \mathbf{C}_{KP} of the new equation:

$$\mathbf{C}_{KP} = \frac{C_p}{2} \left(\frac{(1+r) + C_p^2(1-r)(K_x/\omega)^2 + (K_y/K_x)^2}{r + C_p^2(1-r)(K_x/\omega)^2} \right) \left(\frac{K_x}{K^2} \right) \mathbf{K}, \quad (15)$$

where $K^2 = K_x^2 + K_y^2$ denotes the magnitude square of the directional wave number vector \mathbf{K} . The magnitude of the phase celerity $|\mathbf{C}_{KP}|$ divided by the shallow water celerity C_s is

$$\frac{C_{KP}}{C_s} = \frac{1}{2} \left(\frac{(1+r) + (1-r)(K_x h/kh)^2 + (K_y/K_x)^2}{r + (1-r)(K_x h/kh)^2} \right) \sqrt{\frac{\tanh kh}{kh(1 + (K_y/K_x)^2)}}, \quad (16)$$

where use has been made of $\omega = kC_p$. The wave frequency specified for the wave model, ω , may or may not be the same as the incident wave frequency Ω . When $\omega = \Omega$ then $kh = K_x h$ and for unidirectional case $K_y = 0$ the linear dispersion and shoaling characteristics of the wave equation exactly matches with the linear theory, $C_{KP} = C_p$. Linear theory dispersion relationship for directional waves with wave number K is

$$\frac{C_{EX}}{C_s} = \sqrt{\frac{\tanh Kh}{Kh}}. \quad (17)$$

For random waves with frequencies different from the specified model frequency the match with linear theory is not exact anymore but quite acceptable in a narrow band of frequencies and for relatively small angles as demonstrated in Figure 1 for directional waves by comparing C_{EX}/C_s of linear theory with C_{KP}/C_s for four different direction angles. The specified model frequency ω is selected such that $kh = \pi/2$ (intermediate water waves) and K_y/K_x is assigned to $\tan \theta$ while Kh is varied over the range $0 - 2\pi$. Note that for calculating C_{KP}/C_s as a function of Kh , $K^2 = K_x^2 + K_y^2$ is employed so that $K_x h = Kh / \sqrt{1 + (K_y/K_x)^2}$ which is the required variable in Equation (16). As seen in Figure 1, the dispersion relation of the new KP model C_{KP}/C_s does not converge to unity when Kh tends to zero. There are two different reasons to this; the first reason originates from the characteristics of the wave model of Nadaoka, Beji, and Nakagawa (1997). The dispersion relationship of this particular wave model makes a tangential contact with the exact dispersion relation at the point Kh which corresponds to the specified kh value of the wave model. In the present example, ω for the wave model is selected such that $kh = \pi/2$. Therefore,

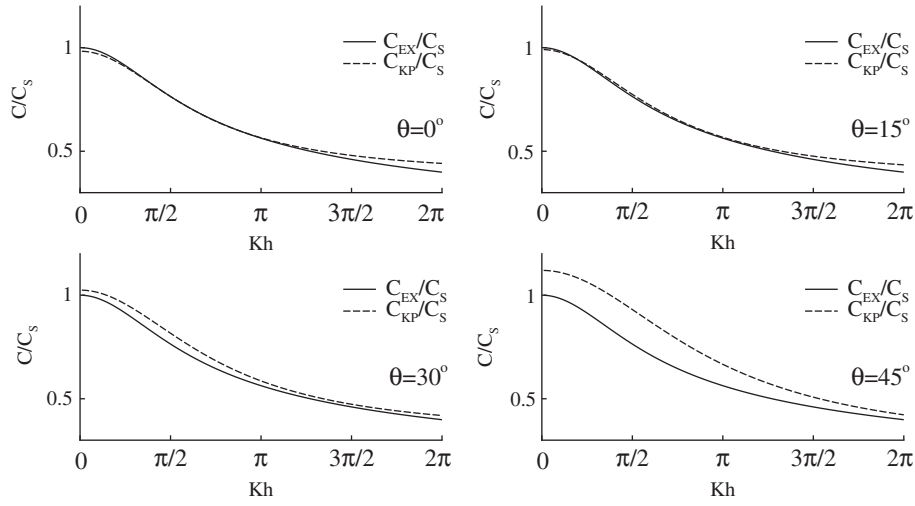


Figure 1. Linear dispersion relationship of the new KP type equation for directional propagation compared with linear theory for a range of Kh values. Four different propagation angles $\theta = 0^\circ, 15^\circ, 30^\circ$, and 45° shown for a model frequency ω specification corresponding to $kh = \pi/2$ intermediate water waves.

for $\theta = 0^\circ$ the dispersion relation of the model is exact at $Kh = kh = \pi/2$ but slowly diverges as Kh goes in either direction: $Kh \rightarrow 0$ or $Kh \rightarrow \infty$. On the other hand, the dispersion relation of classical KP equation is different. It is an asymptotic expansion of the exact relation in the close vicinity of long wave limit $Kh \rightarrow 0$ but diverges relatively rapidly as Kh gets larger.

The second reason is the effect of wave directionality. As the wave angle θ increases the dispersion relationship, Equation (16), diverges from the exact form due to the presence of $(K_y/K_x)^2$ term in the numerator. This divergence is seen to be quite pronounced for $\theta = 45^\circ$ case of Figure 1. For exactly the same reason exactly the same behavior is observed for the dispersion relation of the classical KP equation when the wave angle θ increases. Indeed, directional wave behavior of the dispersion relationship for various wave angles demonstrated in Figure 1 serves as archetype for any kind of KP equation.

4. Numerical scheme

Feng and Mitsui (1998) proposed an implicit finite-difference scheme based on a Crank–Nicolson type formulation for numerical solutions of the original KdV and KP equations. In the same vein, Mekki and Ali (2013) employed a Crank–Nicolson discretization for the solution of a KP equation derived from the BBM equation of Benjamin, Bona, and Mahony (1972). Finite-difference schemes are preferable for practical applications therefore the new KP equation is discretized by a Crank–Nicolson type implicit formulation adapted from Feng and Mitsui (1998). While the present KP equation is quite different from the one considered by Feng and Mitsui (1998) the finite-difference formulation follows their scheme closely with the

exception of ζ_{yy} term, which is treated here in uncoupled manner with appreciable improvement in computational efficiency.

4.1. Finite-difference representation of new KP equation

A Crank–Nicolson type finite-difference discretization of Equation (12) results in

$$\begin{aligned}
 & d_x(\zeta_{ij}^{k+1} - (\zeta_{ij}^{k+1} \zeta_{ij}^k)) \\
 & + \frac{2\Delta t \Delta x}{A_{ij} \delta_x^2(\zeta_{ij}^{k+1} + (\zeta_{ij}^{k+1} \zeta_{ij}^k))} \\
 & - B_{ij} \frac{d_x \delta_x^2(\zeta_{ij}^{k+1} - (\zeta_{ij}^{k+1} \zeta_{ij}^k))}{2\Delta t \Delta x^3} \\
 & - C_{ij} \frac{\delta_x^4(\zeta_{ij}^{k+1} + (\zeta_{ij}^{k+1} \zeta_{ij}^k))}{2\Delta x^4} \\
 & + D_{ij} \frac{\delta_x^2(f_{ij}^{k+1} + f_{ij}^k)}{2\Delta x^2} \\
 & + E_{ij} \frac{d_x(\zeta_{ij}^{k+1} \zeta_{ij}^k)}{2\Delta x} \\
 & - F_{ij} \frac{\delta_x^2(\zeta_{ij}^{k+1} - (\zeta_{ij}^{k+1} \zeta_{ij}^k))}{\Delta t \Delta x^2} \\
 & - G_{ij} \frac{d_x \delta_x^2(\zeta_{ij}^{k+1} \zeta_{ij}^k)}{2\Delta x^3} \\
 & + H_{ij} \frac{\delta_y^2(\zeta_{ij}^{k+1} + (\zeta_{ij}^{k+1} \zeta_{ij}^k))}{2\Delta y^2} = 0
 \end{aligned} \tag{18}$$

where the coefficients A_{ij}, \dots, H_{ij} , etc. correspond to those in Equation (12) in respective order. Note that all the coefficients are spatially varying quantities and computed at each grid point. The centered difference operators are defined as $d_x \zeta_{ij}^k = \zeta_{i+1,j}^k - \zeta_{i-1,j}^k$, $\delta_x^2 \zeta_{ij}^k = \zeta_{i+1,j}^k - 2\zeta_{ij}^k + \zeta_{i-1,j}^k$, $d_x \delta_x^2 \zeta_{ij}^k = \zeta_{i+2,j}^k - 2\zeta_{i+1,j}^k + 2\zeta_{i-1,j}^k - \zeta_{i-2,j}^k$, and $\delta_x^4 \zeta_{ij}^k =$

$\zeta_{i+2}^k - 4\zeta_{i+1,j}^k + 6\zeta_{i,j}^k - 4\zeta_{i-1,j}^k + \zeta_{i-2,j}^k$ with i and j denoting the indices multiplying the spacings Δx and Δy between the grid points in the x – and y – directions, respectively. $\delta_y^2 \zeta_{i,j}^k$ is the y – direction analogy of $\delta_x^2 \zeta_{i,j}^k$. Superscript k stands for the time level index and gives the actual time of simulation when multiplied by the time increment Δt . The non-linear function $f_{i,j}^k$ is defined as $f_{i,j}^k = \zeta_{i,j}^k \zeta_{i,j}^k$ and the summation $f_{i,j}^{k+1} + f_{i,j}^k$ is expressed in a semi-linear form $f_{i,j}^{k+1} + f_{i,j}^k = 2\zeta_{i,j}^{k+1} \zeta_{i,j}^k$ as in Feng and Mitsui (1998) for carrying out the implicit formulation without necessity of iteration due to the non-linear term. The formal approximation procedure may be found in Feng and Mitsui (1998); the informal way of expressing $\zeta_{i,j}^{k+1} \zeta_{i,j}^{k+1} + \zeta_{i,j}^k \zeta_{i,j}^k$ as $2\zeta_{i,j}^{k+1} \zeta_{i,j}^k$ may simply be accomplished by employing the approximation that $(\zeta_{i,j}^{k+1} - \zeta_{i,j}^k)^2 \simeq 0$ hence $\zeta_{i,j}^{k+1} \zeta_{i,j}^{k+1} + \zeta_{i,j}^k \zeta_{i,j}^k \simeq 2\zeta_{i,j}^{k+1} \zeta_{i,j}^k$.

Arranging Equation (18) by placing the unknown new time level terms on the left and the known previous time level values and the complete ζ_{yy} discretization on the right results in a penta-diagonal matrix equation. A penta-diagonal system may be solved by four sweeps: the first two sweeps reduce the system to a tri-diagonal matrix and the next two sweeps solve the tri-diagonal system. In the process employed here the new time level values contained in ζ_{yy} on the right are treated as known. However, in the first iteration only the old time level values are used for computing ζ_{yy} ; the new time level values are introduced in subsequent iterations. For numerical stability, this treatment technique is found to be crucial. The uncoupled treatment of ζ_{yy} necessarily requires iteration; only three iterations are observed to be sufficient for quite satisfying accuracy for all the cases presented here.

4.2. Treatment of boundaries

Test cases and practical applications usually use an incoming boundary across which incident wave field is introduced and an outgoing boundary where waves are radiated away outside the computational domain. Lateral boundaries are generally taken as impermeable side walls with the so-called mirror condition, $\zeta_y = 0$, which is relatively simpler to implement. Specification of incoming waves at the first node of the domain is trivial; the prescribed incident wave form is assigned numerically to the new time level surface elevation $\zeta_{1,j}^{k+1}$ of the first node at each time step. Normally, the second and following nodes should be computed from the discretized wave equation. Wave equations with spatially second-order derivatives result in tri-diagonal matrix systems and pose no problems in this aspect. However, in the present

case the presence of both third and fourth spatial derivatives gives rise to a penta-diagonal system, as indicated before. This problem may be overcome either by one-sided discretization of these higher-order derivatives or by simplifying the wave equation itself by appropriate means. One-sided discretization showed numerical instabilities and was abandoned. Instead, Equation (12) has been simplified by the use of the zeroth-order identities $\zeta_{xx} = -k^2 \zeta$ and $\zeta_{xt} = k\omega \zeta$ in the dispersion and shoaling terms so that the following equation has been used for the second, third, and the $(n - 1)$ nodes.

$$\begin{aligned} \zeta_{xt} + C_p \zeta_{xx} + \frac{ar}{2C_p} (\zeta^2)_{xx} + \frac{1}{2} [(1 + 2r)C_{px} \\ + C_p r_x] \zeta_x + \frac{C_p}{2} \zeta_{yy} = 0 \end{aligned} \quad (19)$$

The last node in the x – direction, n , requires further care by backward differentiation of the second derivative. Frequently a radiation boundary condition is implemented on the outgoing boundary. Such a condition, being a further simplified and manipulated form of Equation (19), is used simply because the full wave equation cannot be discretized appropriately at or near the last nodal point. Ideally, it should be the wave equation itself propagating the waves out without any interruption. Pursuing such an idea for Boussinesq equations and comparing the results with those obtained from the use of simple linear Sommerfeld radiation condition Kiyokawa, Nadaoka, and Beji (1996) showed the remarkable advantage of using only the wave equations in preventing artificially reflecting waves. In the present case, the KdV version of Equation (19) with only first-order derivatives and without $\frac{1}{2} C_p \zeta_{yy}$ term is deemed suitable and used.

5. Simulations

Sample simulations are presented for exploring the capabilities of the newly derived KP type equation. First, unidirectional wave propagation over a sinusoidally varying bottom topography is considered for intermediate and short waves. Simulations are depicted against theoretical wave envelopes drawn according to the energy flux concept. Second, similar to the cnoidal wave patterns experimentally produced by Hammack, Scheffner, and Segur (1989), genuinely two-dimensional obliquely intersecting cnoidal and Stokes waves are produced numerically. The remarkable point in these simulations is the ability of the new KP model to produce not only shallow water cnoidal waves but also deep water Stokes waves. Finally, non-linear refraction-diffraction of waves over a converging zone for three different periods are simulated and corresponding harmonic amplitudes are compared with the measurements of Whalin (1971).

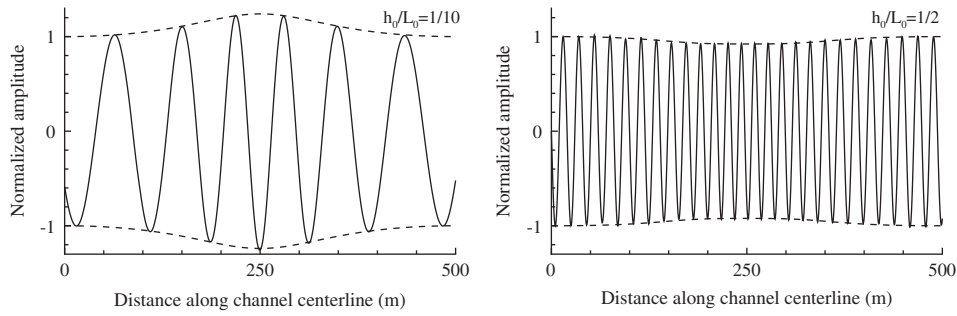


Figure 2. Amplitude variation over a sinusoidally varying bathymetry. Envelopes (dashed lines) are computed from the constancy of energy flux. Left: Long to intermediate incident waves, $h_0/L_0 = 1/10$. Right: Short incident waves, $h_0/L_0 = 1/2$.

6. 1-D linear waves over varying bathymetry

While accurate prediction of wave heights is an essential engineering requirement that would be improved much via a reliable shoaling model, Stiassnie (2017), based on the work of Bonnefoy et al. (2016), concludes that weakly non-linear theories are satisfactory enough to model non-linearities. As the inclusion of higher-order non-linearity is not crucial in practical applications the present work places particular emphasis on the accuracy of linear shoaling prediction and presents here two sample cases of linear wave propagation over a sinusoidally varying depth. The water depth is initially $h_0 = 10$ m, reduces to $h_m = h_0/3 = 3.33$ m at mid-length of channel and then increases to $h_0 = 10$ m again. The wave period is adjusted such that at the channel entrance and exit $h_0/L_0 = 1/10$ for the first case (long to intermediate incident waves) and $h_0/L_0 = 1/2$ for the second case (short incident waves). The incident wave amplitude is set arbitrarily to $a_0 = 1$ m as the simulation is performed with linearized equation. Figure 2 shows for both cases the performance of the new KP type equation against the wave envelope drawn according to the energy flux concept $a^2 C_g = \text{Const.}$ with C_g taken from the exact linear theory. The agreement with theory for both cases is nearly perfect; such an agreement confirms the reliability of the present model in computing wave amplitude variations over varying bathymetry for entire range of relative depths.

7. 2-D cnoidal and stokes waves

In order to demonstrate that the new KP equation is capable of simulating shallow water cnoidal and deep

water Stokes waves equally well, two numerical simulations of obliquely intersecting wave trains are performed. For cnoidal waves, the period is $T = 2.89$ s, the wave height to water depth ratio $H/h = 0.05$, and the elliptic parameter $m = 1 - 0.215$, as selected from the laboratory experiments (CN2) of Goring and Reichlen (1980). For Stokes waves, the period is $T = 3.57$ s and the non-linearity parameter $k_5 H = 0.15$ with k_5 denoting the wave number according to the third-order Stokes theory. At the incident boundary along the y – axis two separate wave trains directed at angles $+\theta$ and $-\theta$ to the x – axis were generated and superposed. This superposition doubled the values of non-linearity parameters of the incident wave fields for both cases. Generation of a wave train with an angle to the x – axis was realized by introducing a time phase lag of $\Delta t_l = j\Delta y \sin \theta / c$ in the argument of the wave function at each y – node $j = 0, \dots, m$. This method is exactly the same as the one used by Hammack, Scheffner, and Segur (1989) in their laboratory experiments. The directed wave angle was taken as $\theta \simeq 19.5^\circ$ so that the directed wavelength in y – direction was three times the generating wavelength $L_y = L / \sin \theta = 3L$ for both cases while $L = 4$ m, $h/L = 1/20$ (shallow water) for cnoidal waves and $L = 20$ m, $h/L = 1/2$ (deep water) for Stokes waves. Again for both cases the time resolution was $\Delta t = T/60$ s and the spatial resolution in the propagation direction was adjusted to give a Courant number that was nearly unity. Due to numerical stability problems for $\Delta y \sim \Delta x$ the resolution in the y – direction was several times coarser in comparison with the x – direction. The simulation of deep water Stokes waves was more sensitive in this aspect and required even coarser transverse

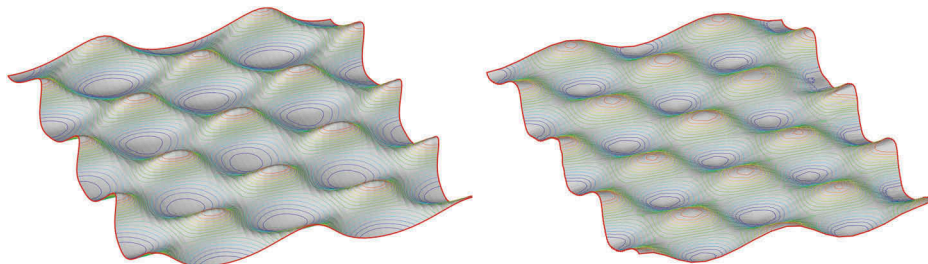


Figure 3. Perspective views of obliquely intersecting cnoidal waves (left) and Stokes waves (right).

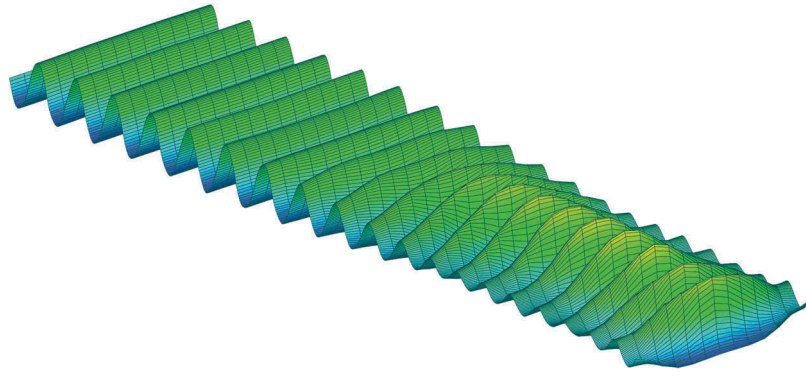


Figure 4. A perspective view of the fully developed numerical wave tank for Whalin's experiment of $T = 1$ s waves.

grids. Figure 3 shows perspective views of fully developed cnoidal and Stokes wave fields.

7.1. Non-linear waves over a topographical lens

The last case considers non-linear wave convergence over a bottom topography that acts as a focusing lens (Whalin, 1971). The physical wave tank used in the experiments was $84 \text{ ft} = 25.6 \text{ m}$ long and $20 \text{ ft} = 6.096 \text{ m}$ wide. In the middle part of the wave tank, 11 semicircular steps were evenly spaced to form a topographical lens. The equations defining the bottom are given in Whalin (1971). Experiments were carried out by generating regular sinusoidal waves with periods $T = 1, 2$, and 3 s . Primary wave and harmonic amplitudes along the centerline of the wave tank were obtained at various stations.

Figure 4 shows the perspective view of the numerical wave tank after 38 periods of simulation. The converged and diverged regions of wave forms and the relatively shortened length of the waves in the shallow region near the end of the domain are notable features of the simulation. For all three cases, the simulations were performed with a span-wise resolution Δy of $1/12$ of the wave tank width. For incident wave period $T = 1 \text{ s}$, the incident wave amplitude is $a_0 = 1.95 \text{ cm}$ in water depth of $h_0 = 0.4572 \text{ m}$. The time-step and the x -direction resolution were respectively $\Delta t = T/50 \text{ s}$ and $\Delta x = L_m/50 \text{ m}$ with $L_m = 1.3 \text{ m}$ denoting the mean wavelength computed as the average of the deep water $h_0 = 1.5 \text{ ft} = 0.4572 \text{ m}$ and shallow water $h_s = 0.5 \text{ ft} = 0.1524 \text{ m}$ wavelengths, which are $L_0 = 1.5 \text{ m}$ and $L_s = 1.1 \text{ m}$. Note that this case is very close to deep water conditions since $h_0/L_0 = 1/3$.

Figure 5 top graph depicts the measured data and the computed results for the primary wave and the first harmonic amplitudes for $T = 1 \text{ s}$ waves. In the middle the case for $T = 2 \text{ s}$ and $a_0 = 0.75 \text{ cm}$ is shown for the primary wave and two harmonics. The bottom graph makes the same

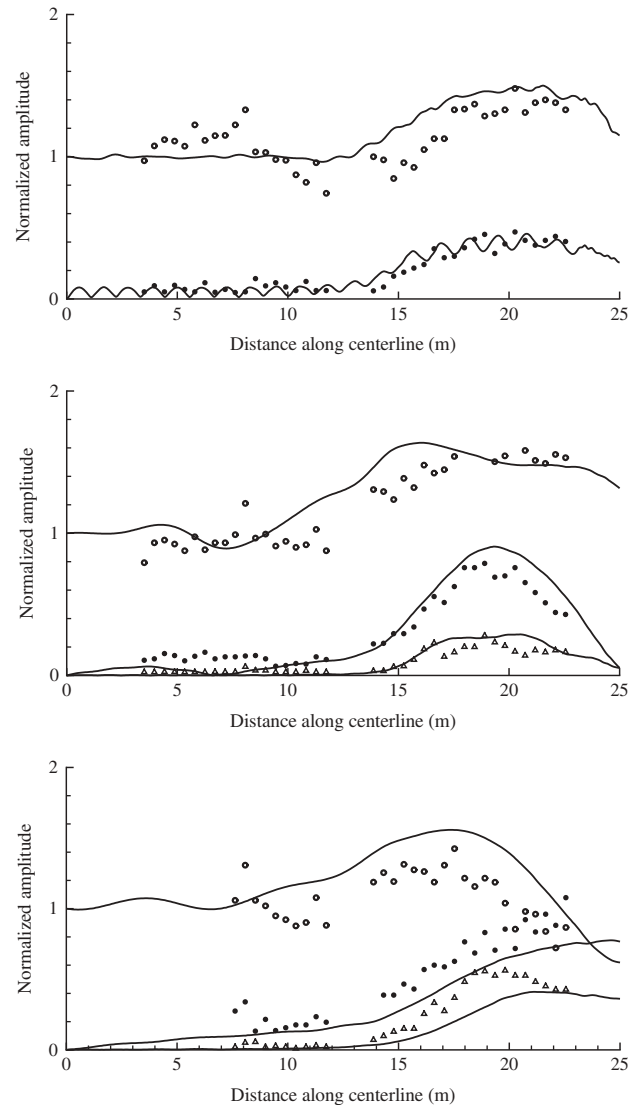


Figure 5. Top: Whalin's experiments for incident waves of $T = 1 \text{ s}$. Middle: Incident waves of $T = 2 \text{ s}$. Bottom: Incident waves of $T = 3 \text{ s}$. Measured and computed harmonic amplitudes along the centerline of the wave tank. Solid line: computation, scatter: experimental data.

comparisons for $T = 3 \text{ s}$ and $a_0 = 0.68 \text{ cm}$. All the computations were carried out with $\Delta t = T/50$ and $\Delta x = L_m/50$.

8. Concluding remarks

A Kadomtsev–Petviashvili type equation for the entire range of relative water depths has been derived. The newly derived equation is capable of simulating shallow and deep water waves equally well. Linear shoaling characteristics, which are essential in accurate wave height calculations, are tested for relatively long and short waves propagating over varying water depths with very satisfactory results. The new wave model may be used for accurate estimation of wave conditions for waves propagating from deep to shallow regions.

Disclosure statement

No potential conflict of interest was reported by the author.

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References

- Beji, S., and K. Nadaoka. 1997. "A Time-Dependent Nonlinear Mild-Slope Equation for Water Waves." *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 453: 319–332. doi:10.1098/rspa.1997.0018.
- Benjamin, T. B., J. L. Bona, and J. J. Mahony. 1972. "Model Equations for Long Waves in Nonlinear Dispersive Systems." *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences* 272: 47–78. doi:10.1098/rsta.1972.0032.
- Berkhoff, J. C. W. 1972. "Computation of Combined Refraction-Diffraction." *Proceedings 13th International Conference on Coastal Engineering* 1: 471–490.
- Bonnefoy, F., F. Haudin, G. Michel, B. Semin, T. Humbert, S. Aumaitre, M. Berhanu, and E. Falcon. 2016. "Observation of Resonant Interactions among Surface Gravity Waves." *Journal Fluid Mechanisms* 805 (R3): 1–12. doi:10.1017/jfm.2016.576.
- Boussinesq, J. V. 1872. "Theory of Waves and Surges Which Propagate the Length of a Horizontal Rectangular Canal, Imparting to the Fluid Contained within the Canal Velocities that are Sensibly the Same from the Top to the Bottom." *Journal Mathematical Pures and Applications* 17: 55–108. (Translated by A. C. J. Vastano and J. C. H. Mungall, March 1976.).
- Bryant, P. J. 1982. "Two-Dimensional Periodic Permanent Waves in Shallow Water." *Journal Fluid Mechanisms* 115: 525–532. doi:10.1017/S0022112082000895.
- Feng, B.-F., and T. Mitsui. 1998. "A Finite Difference Method for the Korteweg-de Vries and the Kadomtsev-Petviashvili Equations." *Journal of Computational and Applied Mathematics* 90: 95–116. doi:10.1016/S0377-0427(98)00006-5.
- Goring, D., and F. Reichlen. 1980. "The Generation of Long Waves in the Laboratory." *Proceedings 17th International Conference on Coastal Engineering* 1: 763–783.
- Hammack, J., N. Scheffner, and H. Segur. 1989. "Two-Dimensional Periodic Waves in Shallow Water." *Journal Fluid Mechanisms* 209: 567–589. doi:10.1017/S0022112089003228.
- Isobe, M. 1994. "Time-Dependent Mild-Slope Equations for Random Waves." *Proceedings 24th International Conference on Coastal Engineering* 1: 285–299.
- Kadomtsev, B. B., and V. I. Petviashvili. 1970. "On the Stability of Solitary Waves in Weakly Dispersive Media." *Soviet Physical Doklady* 15: 539–541.
- Kiyokawa, T., K. Nadaoka, and S. Beji. 1996. "An Open-Boundary Treatment for Simulation of Nonlinear Wave Propagation." *Proceedings Coastal Engineering, JSCE* 43-1: 1–5. in Japanese.
- Kodama, Y. 2010. "KP Solitons in Shallow Water." *Journal of Physics A: Mathematical and Theoretical* 43: 434004–434054. doi:10.1088/1751-8113/43/43/434004.
- Korteweg, D. J., and G. de Vries. 1895. "XLI. On the Change of Form of Long Waves Advancing in a Rectangular Canal, and on a New Type of Long Stationary Waves." *Philosophical Magazine Series* 5 39: 422–443. doi:10.1080/14786449508620739.
- Lamb, H. 1932. *Hydrodynamics*. New York: Dover Publications.
- Li, W., Yeh, H. & Kodama, Y. 2011. "On the Mach Reflection of a Solitary Wave: Revisited". *Journal Fluid Mechanisms* 672: 326–357. doi:10.1017/S0022112010006014.
- Luke, J. C. 1967. "A Variational Principle for a Fluid with a Free Surface." *Journal Fluid Mechanisms* 27: 395–397. doi:10.1017/S0022112067000412.
- Madsen, O. S., and C. C. Mei. 1969. "The Transformation of a Solitary Wave over an Uneven Bottom." *Journal Fluid Mechanisms* 39 (4): 781–791. doi:10.1017/S0022112069002461.
- Mei, C. C., and B. Le Méhauté. 1966. "Note on the Equations of Long Waves over an Uneven Bottom." *Journal of Geophysical Research* 71 (2): 393–400. doi:10.1029/JZ071i002p00393.
- Mekki, A., and M. M. Ali. 2013. "Numerical Simulation of Kadomtsev–Petviashvili–Benjamin–Bona–Mahony Equations Using Finite Difference Method." *Applied Mathematics and Computation* 219: 11214–11222. doi:10.1016/j.amc.2013.04.039.
- Nadaoka, K., S. Beji, and Y. Nakagawa. 1994. "A Fully-Dispersive Nonlinear Wave Model and Its Numerical Solutions." *Proceedings 24th International Conference on Coastal Engineering* 1: 427–441.
- Nadaoka, K., S. Beji, and Y. Nakagawa. 1997. "A Fully Dispersive Weakly Nonlinear Model for Water Waves." *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 453: 303–318. doi:10.1098/rspa.1997.0017.
- Peregrine, D. H. 1967. "Long Waves on a Beach." *Journal Fluid Mechanisms* 27: 815–827. doi:10.1017/S0022112067002605.
- Stiassnie, M. 2017. "On the Strength of the Weakly Nonlinear Theory for Surface Gravity Waves." *Journal Fluid Mechanisms* 810: 1–4. doi:10.1017/jfm.2016.632.
- Tsuji, H., and M. Oikawa. 2007. "Oblique Interaction of Solitons in an Extended Kadomtsev-Petviashvili Equation." *Journal of the Physical Society of Japan* 76: 84401–84408. doi:10.1143/JPSJ.76.084401.
- Whalin, R. W. 1971. "The Limit of Applicability of Linear Wave Refraction Theory in a Convergence Zone." Res. Rep. H-71-3, U.S. Army Corps of Engrs., Vicksburg, MI: Waterways Expt. Station.
- Yeh, H., W. Li, and Y. Kodama. 2010. "Mach Reflection and KP Solitons in Shallow Water." *The European Physical Journal Special Topics* 185: 97–111. doi:10.1140/epjst/e2010-01241-0.